UDC 621.314.057

https://doi.org/10.33271/nvngu/2021-5/080

M. S. Seheda,
orcid.org/0000-0001-8459-5758,
Ye. V. Cheremnykh,
orcid.org/0000-0002-4621-2426,
P. F. Gogolyuk,
orcid.org/0000-0003-0002-4638,
Yu. V. Blyznak,
orcid.org/0000-0002-4914-2283

Lviv Polytechnic National University, Lviv, Ukraine, e-mail: mykhailo.s.seheda@lpnu.ua

MATHEMATICAL MODELING OF WAVE PROCESSES IN TWO-WINDING TRANSFORMERS TAKING INTO ACCOUNT THE MAIN MAGNETIC FLUX

Purpose. To create a method for mathematical modeling of wave processes in power two-winding transformers based on a substitute scheme, which takes into account the design features of power transformers.

Methodology. Formation of mathematical models for the research on wave processes in power two-winding transformers and further development of the analytical method for solving the system of partial differential equations.

Findings. A mathematical model for the research on wave processes in power two-winding transformers based on a substitution scheme, witch adequately takes into account both electrical and magnetic connections, is created and an improved analytical method is proposed for solving a system of partial differential equations which allows taking into account the interval time of propagation of electromagnetic waves along the entire length of the windings and the time interval, during which the voltage changes significantly from its complete change during the wave processes,.

Originality. The paper proposes a mathematical model for the research on wave processes in the windings of power two-winding transformers based on its alternative scheme, which takes into account electrical and magnetic connections, and improves the Fourier method for solving a system of differential equations with partial derivatives.

Practical value. A mathematical model is created for calculating wave processes in transformers, which allows analyzing the voltage distribution in the transformer windings during the action of pulse voltage on them and adjusting their insulating abilities, given that the operation of power transformers is subject to high requirements for the reliability of their work.

Keywords: *wave process, mathematical model, transformer, differential equations in partial derivatives, boundary value problem*

Introduction. The influence of impulse overvoltage on the insulation of the transformers requires correct coordination of insulation, which has defined value during exploitation. Generally, to ensure the proper adequacy of the results of mathematical modeling, it is necessary to take into account the electromagnetic connections between the transformer windings and the main magnetic flux. Nowadays, taking into account the above factors, the creation of mathematical models for the research of wave processes in transformers is relevant.

Literature review. The research on wave process was for a long time carried out on the basis of the substitution scheme of one winding without taking into account the electromagnetic connections and the interconnections between the windings. At present, methods of analysis of wave processes in transformer windings are directed at creating mathematical models, taking into account electromagnetic connections between transformer windings [1–4]. Modeling by the method of “white box” requires the formation of mathematical elements of the power system, taking into account all the parameters of the replacement circuit of the element, which allows researching its internal periodic processes [5, 6]. The calculation of the parameters of the replacement circuit of the transformer, with the assumption of a linear voltage distribution along the winding and the internal transients, by ordinary differential equations is described [7, 8]. This approach does not allow researching the wave processes in the windings of transformers. Peculiarities of resonant overvoltages, without taking into account mutual inductive connections between windings of the winding, for one transformer winding of different types are considered [9, 10]. To solve the equations of partial derivatives of Laplace transforms and transfer functions without taking into account the mutual inductive connections between the windings is proposed in [11, 12].

To research the processes in transformers during the action of external overvoltages, there is a waveform provided in [1].

The purpose of the proposed article is to create a mathematical model for the research on wave processes in two-winding transformers taking into account the electromagnetic connections between windings and turns of windings, solving the obtained differential-integral equations in partial derivatives using Fourier series.

Main material and mathematical model of wave processes in two-winding transformer. A mathematical model for the research on wave processes in transformers with two-windings, taking into account the electromagnetic connections between the windings and turns of the windings, is created basing on the subschema given in Fig. 1. [2]. The obtained differential-integral equations are solved using partial derivatives and the Fourier series.

The equation of change in currents flowing through the windings is written basing on Kirchhoff’s current law (*1st* law).

$$\frac{-\partial i_1(x,t)}{\partial x} = g_{01}u_1(x,t) + (C_{01} + C_{012})\frac{\partial u_1(x,t)}{\partial t} - \\ - C_{012}\frac{\partial u_2(x,t)}{\partial t} - C_{M01}\frac{\partial^3 u_1(x,t)}{(\partial x^2 \partial t)}; \quad (1)$$

$$\frac{-\partial i_2(x,t)}{\partial x} = g_{02}u_2(x,t) + (C_{02} + C_{012})\frac{\partial u_2(x,t)}{\partial t} - \\ - C_{012}\frac{\partial u_1(x,t)}{\partial t} - C_{M02}\frac{\partial^3 u_2(x,t)}{(\partial x^2 \partial t)}. \quad (2)$$

The equation of voltage fall per unit length of windings is written basing on Kirchhoff’s voltage law (*2st* law).

$$-\frac{\partial u_1(x,t)}{\partial x} = r_{01}i_1(x,t) + L_{01}\frac{\partial i_1(x,t)}{\partial t} + M_0\frac{\partial i_2(x,t)}{\partial t} + \\ + \int_0^x M_1(x,s)\frac{\partial i_1(x,t)}{\partial t}ds + \int_x^l M_1(x,s)\frac{\partial i_1(x,t)}{\partial t}ds; \quad (3)$$

$$-\frac{\partial u_2(x,t)}{\partial x} = r_{02}i_2(x,t) + L_{02}\frac{\partial i_2(x,t)}{\partial t} + M_0\frac{\partial i_1(x,t)}{\partial t} + \\ + \int_0^x M_2(x,s)\frac{\partial i_2(x,t)}{\partial t}ds + \int_x^l M_2(x,s)\frac{\partial i_2(x,t)}{\partial t}ds. \quad (4)$$

In equation (3, 4) $L_{01}=L_{\mu 0}+L_{\sigma 01}$, $M_0=\dfrac{L_{\mu 0}}{k}+M_{\sigma 0}$, $L_{02}=$ $=\dfrac{L_{\mu 0}}{k^2}+L_{\sigma 02}$, $L_{\sigma 10}$, $L_{\sigma 20}$, $M_{\sigma 0}$ are self and mutual scattering inductances of the primary and secondary windings and between them; $L_{\mu 0}$ is inductance of the magnetic system of the transformer; $M_1(x, s)$, $M_2(x, s)$ are self and mutual inter-turn inductances of scattering of primary and secondary windings; k is transformer ratio; l is a length of windings; x is instant longitudinal coordinate, s is the instant coordinate, which determines the distance from place x to the coordinate of any other place on the axis of the winding.

Equations (1, 2) have differentiated by t

$$-\frac{\partial^2 i_1(x,t)}{\partial x\partial t}=g_{01}\frac{\partial u_1(x,t)}{\partial t}+(C_{01}+C_{012})\frac{\partial^2 u_1(x,t)}{\partial t^2}-C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}-C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}; \tag{5}$$

$$-\frac{\partial^2 i_2(x,t)}{\partial x\partial t}=g_{02}\frac{\partial u_2(x,t)}{\partial t}+(C_{02}+C_{012})\frac{\partial^2 u_2(x,t)}{\partial t^2}-C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}-C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}. \tag{6}$$

Equations (3, 4) are differentiated by x, in order to obtain next equations

$$-\frac{\partial^2 u_1(x,t)}{\partial x^2}=r_{01}\frac{\partial i_1(x,t)}{\partial x}+L_{01}\frac{\partial^2 i_1(x,t)}{\partial x\partial t}+M_0\frac{\partial^2 i_2(x,t)}{\partial x\partial t}+\int_0^x\left(M_1(x,s)\frac{\partial^2 i_1(x,t)}{\partial x\partial t}+\frac{\partial M_1(x,s)}{\partial x}\frac{\partial i_1(x,t)}{\partial t}\right)ds+\int_x^l\left(M_1(x,s)\frac{\partial^2 i_1(x,t)}{\partial x\partial t}+\frac{\partial M_1(x,s)}{\partial x}\frac{\partial i_1(x,t)}{\partial t}\right)ds; \tag{7}$$

$$-\frac{\partial^2 u_2(x,t)}{\partial x^2}=r_{02}\frac{\partial i_2(x,t)}{\partial x}+L_{02}\frac{\partial^2 i_2(x,t)}{\partial x\partial t}+M_0\frac{\partial^2 i_1(x,t)}{\partial x\partial t}+\int_0^x\left(M_2(x,s)\frac{\partial^2 i_2(x,t)}{\partial x\partial t}+\frac{\partial M_2(x,s)}{\partial x}\frac{\partial i_2(x,t)}{\partial t}\right)ds+\int_x^l\left(M_2(x,s)\frac{\partial^2 i_2(x,t)}{\partial x\partial t}+\frac{\partial M_2(x,s)}{\partial x}\frac{\partial i_2(x,t)}{\partial t}\right)ds. \tag{8}$$

Equations (1, 2) and (5, 6) are substituted in (9, 10); as a result, obtain

$$\begin{aligned}\frac{\partial^2 u_1(x,t)}{\partial x^2}&=r_{01}(-g_{01}u_1(x,t)-(C_{01}+C_{012})\frac{\partial u_1(x,t)}{\partial t}+C_{012}\frac{\partial u_2(x,t)}{\partial t}+C_{M01}\frac{\partial^3 u_1(x,t)}{\partial^2 x\partial t}+L_{01}\Big(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-\\&-(C_{01}+C_{012})\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}+M_0\Big(-g_{02}\frac{\partial u_2(x,t)}{\partial t}-(C_{02}+C_{012})\frac{\partial^2 u_2(x,t)}{\partial t^2}+\\&+C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}+\int_0^x M_1(x,s)\Big(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-(C_{01}+C_{012})\frac{\partial^2 u_1(x,t)}{\partial t^2}+\\&+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Big)+\frac{\partial M_1(x,s)}{\partial x}\frac{\partial i_1(x,t)}{\partial t}\Big)ds+\int_x^l M_1(x,s)\Big(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-(C_{01}+C_{012})\frac{\partial^2 u_1(x,t)}{\partial t^2}+\\&+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Big)+\frac{\partial M_1(x,s)}{\partial x}\frac{\partial i_1(x,t)}{\partial t}\Big)ds;\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial^2 u_2(x,t)}{\partial x^2}&=r_{02}(-g_{02}u_2(x,t)-(C_{02}+C_{012})\frac{\partial u_2(x,t)}{\partial t}+C_{012}\frac{\partial u_1(x,t)}{\partial t}+C_{M02}\frac{\partial^3 u_2(x,t)}{\partial^2 x\partial t}+L_{02}\Big(-g_{02}\frac{\partial u_2(x,t)}{\partial t}-\\&-(C_{02}+C_{012})\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}+M_0\Big(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-\\&-(C_{01}+C_{012})\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}+\int_0^x M_2(x,s)\Big(g_{02}\frac{\partial u_2(x,t)}{\partial t}+\\&+(C_{02}+C_{012})\frac{\partial^2 u_2(x,t)}{\partial t^2}-C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}-C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}\Big)+\frac{\partial M_2(x,s)}{\partial x}\frac{\partial i_2(x,t)}{\partial t}\Big)ds+\\&+\int_x^l M_2(x,s)\Big(g_{02}\frac{\partial u_2(x,t)}{\partial t}+(C_{02}+C_{012})\frac{\partial^2 u_2(x,t)}{\partial t^2}-C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}-C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}\Big)+\frac{\partial M_2(x,s)}{\partial x}\frac{\partial i_2(x,t)}{\partial t}\Big)ds.\end{aligned} \tag{10}$$

We introduce the following notation for equation (9), namely

$a_{11}=r_{01}g_{01}$; $b_{11}=r_{01}(C_{01}+C_{012})$; $c_{11}=r_{01}C_{012}$;

$a_{12}=L_{01}g_{01}$; $b_{12}=L_{01}(C_{01}+C_{012})$; $c_{11}=L_{01}C_{012}$;

$a_{13}=M_0g_{02}$; $b_{13}=M_0(C_{01}+C_{012})$; $c_{13}=M_0C_{012}$;

$a_{14}=g_{01}$; $b_{14}=C_{01}+C_{012}$; $c_{14}=C_{012}$;

$d_{11}=r_{01}C_{M01}$; $d_{12}=L_{01}C_{M01}$; $d_{13}=\mathrm{M}_0C_{M01}$; $d_{14}=C_{M01}$.

As a result, changing the sign in the left and right parts of equation (9), obtain

$$\begin{aligned}\frac{\partial^2 u_1(x,t)}{\partial x^2}&=a_{11}u_1(x,t)+b_{11}\frac{\partial u_1(x,t)}{\partial t}-c_{11}\frac{\partial u_2(x,t)}{\partial t}-\\&-d_{11}\frac{\partial^3 u_1(x,t)}{\partial^2 x\partial t}+a_{12}\frac{\partial u_1(x,t)}{\partial t}+b_{12}\frac{\partial^2 u_1(x,t)}{\partial t^2}-c_{12}\frac{\partial^2 u_2(x,t)}{\partial t^2}-\\&-d_{12}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}+a_{13}\frac{\partial u_2(x,t)}{\partial t}+b_{13}\frac{\partial^2 u_2(x,t)}{\partial t^2}-c_{13}\frac{\partial^2 u_1(x,t)}{\partial t^2}-\\&-d_{13}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}+\int_0^x M_1(x,s)\Big(a_{14}\frac{\partial u_1(x,t)}{\partial t}+b_{14}\frac{\partial^2 u_1(x,t)}{\partial t^2}-\\&-c_{14}\frac{\partial^2 u_2(x,t)}{\partial t^2}-d_{14}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Big)+\frac{\partial M_1(x,s)}{\partial x}\frac{\partial i_1(x,t)}{\partial t}\Big)ds+\\&+\int_x^l M_1(x,s)\Big(a_{14}\frac{\partial u_1(x,t)}{\partial t}+b_{14}\frac{\partial^2 u_1(x,t)}{\partial t^2}-c_{14}\frac{\partial^2 u_2(x,t)}{\partial t^2}-\\&-d_{14}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Big)+\frac{\partial M_1(x,s)}{\partial x}\frac{\partial i_1(x,t)}{\partial t}\Big)ds.\end{aligned} \tag{11}$$

We introduce the following notation for equation (12), namely

$a_{21}=r_{02}g_{02}$; $b_{21}=r_{02}(C_{02}+C_{012})$; $c_{21}=r_{02}C_{012}$;

$a_{22}=L_{02}g_{02}$; $b_{22}=L_{02}(C_{02}+C_{012})$; $c_{22}=L_{02}C_{012}$;

$a_{23}=M_0g_{01}$; $b_{23}=M_0(C_{02}+C_{012})$; $c_{23}=M_0C_{012}$;

$a_{24}=g_{02}$; $b_{24}=C_{02}+C_{012}$; $c_{24}=C_{012}$;

$d_{21}=r_{02}C_{M02}$; $d_{22}=L_{02}C_{M02}$; $d_{23}=\mathrm{M}_0C_{M02}$; $d_{24}=C_{M02}$.

As a result, changing the sign in the left and right parts of equation (10), obtain

$$
\begin{aligned}
\frac{\partial^2 u_2(x,t)}{\partial x^2} &= a_{21}u_2(x,t)+b_{21}\frac{\partial u_2(x,t)}{\partial t}-c_{21}\frac{\partial u_1(x,t)}{\partial t}-\\
&-d_{21}\frac{\partial^3 u_2(x,t)}{\partial x^2\partial t}+a_{22}\frac{\partial u_2(x,t)}{\partial t}+b_{22}\frac{\partial^2 u_2(x,t)}{\partial t^2}-c_{22}\frac{\partial^2 u_1(x,t)}{\partial t^2}-\\
&-d_{22}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}+a_{23}\frac{\partial u_1(x,t)}{\partial t}+b_{23}\frac{\partial^2 u_1(x,t)}{\partial t^2}-c_{23}\frac{\partial^2 u_2(x,t)}{\partial t^2}-\\
&-d_{23}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}+\int_0^x\left(M_2(x,s)\left(a_{24}\frac{\partial u_2(x,t)}{\partial t}+b_{24}\frac{\partial^2 u_2(x,t)}{\partial t^2}-\right.\right.\\
&\left.\left.-c_{24}\frac{\partial^2 u_1(x,t)}{\partial t^2}-d_{24}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}\right)+\frac{\partial M_2(x,s)}{\partial x}\frac{\partial i_2(x,t)}{\partial t}\right)ds+\\
&+\int_x^1\left(M_2(x,s)\left(a_{24}\frac{\partial u_2(x,t)}{\partial t}+b_{24}\frac{\partial^2 u_2(x,t)}{\partial t^2}-c_{24}\frac{\partial^2 u_1(x,t)}{\partial t^2}-\right.\right.\\
&\left.\left.-d_{24}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}\right)+\frac{\partial M_2(x,s)}{\partial x}\frac{\partial i_2(x,t)}{\partial t}\right)ds.
\end{aligned} \tag{12}
$$

Based on physical considerations, the values of subintegral functions in equations (11, 12), for their insignificance, are neglected.

Initial conditions are

$$
\begin{aligned}
&u_1(x,t)|_{t=0}=u_1(x)=U_{m1}-k_1x, \quad x\in(0;l);\\
&\left.\frac{\partial u_1(x,t)}{\partial t}\right|_{t=0}=du_1(x)=0;\\
&u_2(x,t)|_{t=0}=u_2(x)=U_{m2}-k_2x;\\
&\left.\frac{\partial u_2(x,t)}{\partial t}\right|_{t=0}=du_2(x)=0,
\end{aligned} \tag{13}
$$

where U_{m1}, U_{m2} are the voltage amplitude of the primary and secondary windings; k_1, k_2 are the coefficients of the rate of change in voltage along the windings for the moment of time $t=0$.

Boundary conditions are

$$
\begin{aligned}
&u_1(x,t)|_{x=0}=f_{10}(t)=e_{iмn}(t), \quad t>0;\\
&u_1(x,t)|_{x=1}=f_{1l}(t)=0;\\
&u_2(x,t)|_{x=0}=f_{20}(t)=0; \quad u_2(x,t)|_{x=l}=f_{21}(t)=0.
\end{aligned} \tag{14}
$$

Consistency of conditions is

$$
\begin{aligned}
&u_1(x)|_{t=0}=u_1(x,t)|_{t=0}=f_{10}(t)|_{t=0};\\
&u_1(x)|_{x=l}=u_1(x,t)|_{t=0}=f_{11}(t)|_{t=0};\\
&u_2(x)|_{t=0}=u_2(x,t)|_{t=0}=f_{20}(t)|_{t=0};\\
&u_2(x)|_{x=l}=u_2(x,t)|_{t=0}=f_{21}(t)|_{t=0};\\
&f_{10}(0)=u_1(0); \quad f_{1l}(0)=du_1(0);\\
&f_{20}(0)=u_2(0); \quad f_{2l}(0)=du_2(0);
\end{aligned} \tag{15}
$$

$$
\left.\frac{\partial f_{10}(t)}{\partial t}\right|_{t=0}=u_1(0)|_{t=0}; \quad \left.\frac{\partial f_{1l}(t)}{\partial t}\right|_{t=o}=du_1(t);
$$

$$
\left.\frac{\partial f_{20}(t)}{\partial t}\right|_{t=0}=u_2(0)|_{t=0}; \quad \left.\frac{\partial f_{2l}(t)}{\partial t}\right|_{t=0}=du_2(t).
$$

To convert the boundary conditions (15) into null (homogeneous), based on [2], it is expedient to form the solution of level (11) in the following form.

Replacement is performed in the equation (13), namely

$$u_1(x,t)=V_1(x,t)+A_1(t)+xB_1(t). \tag{16}$$

A function $A_1(t)$, $B_1(t)$ is found, which in the replacement of (16) gives $V_1(x, t)$ consistency of conditions (15), namely

$$V_1(x,t)|_{x=0}=0; \quad V_1(x,t)|_{x=l}=0.$$

Then (16) is written for $x=0$ in the following way

$$u_1(x,t)|_{x=0}=V_1(x,t)|_{x=0}+A_1(t)=f_{10}(t). \tag{17}$$

From (17) we obtain

$$A_1(t)=f_{10}(t)=e_{iмn}(t). \tag{18}$$

For $x=l$

$$u_1(x,t)|_{x=l}=V_1(x,t)|_{x=l}+A_1(t)+lB_1(t)=f_{11}(t). \tag{19}$$

From (19) we obtain

$$B_1(t)=\frac{1}{l}(f_{1l}(t)-A_1(t))=-\frac{1}{l}e_{\text{зìï.}}(t). \tag{20}$$

The initial condition for equation (16) acquired the form

$$u_1(x,t)|_{t=0}=V_1(x,t)|_{t=0}+A_1(t)|_{t=0}+xB_1(t)|_{t=0}\equiv u_1(x). \tag{21}$$

From (21) obtain

$$V_1(x,t)|_{t=0}=u_1(x)-A_1(t)|_{t=0}-xB_1(t)|_{t=0}\equiv V_{11}(x). \tag{22}$$

Differentiating (21) by t, obtain

$$\left.\frac{\partial u_1(x,t)}{\partial t}\right|_{t=0}=\left.\frac{\partial V_1(x,t)}{\partial t}\right|_{t=0}+\left.\frac{dA_1(t)}{dt}\right|_{t=0}+x\left.\frac{dB_1(t)}{dt}\right|_{t=0}\equiv du_1(x). \tag{23}$$

From (23) we obtain

$$
\begin{aligned}
\left.\frac{\partial V_1(x,t)}{\partial t}\right|_{t=0}&=du_1(x)-\left.\frac{dA_1(t)}{\partial t}\right|_{t=0}-x\left.\frac{dB_1(t)}{\partial t}\right|_{t=0}=\\
&=-\left.\frac{dA_1(t)}{\partial t}\right|_{t=0}-x\left.\frac{dB_1(t)}{\partial t}\right|_{t=0}\equiv V_{12}(x).
\end{aligned} \tag{24}
$$

The following substitution is made in equation (12)

$$u_2(x,t)=V_2(x,t)+A_2(t)+xB_2(t). \tag{25}$$

A function $A_2(t)$, $B_2(t)$ is found, which in the replacement of (25) gives $V_2(x, t)$ consistency of conditions (15), namely

$$V_2(x,t)|_{x=0}=0; \quad V_2(x,t)|_{x=l}=0.$$

Then (25) for $x=0$ takes next form

$$u_2(x,t)|_{x=0}=V_2(x,t)|_{x=0}+A_2(t)=f_{20}(t). \tag{26}$$

From (26) we obtain

$$A_2(t)=f_{20}(t)=0. \tag{27}$$

For $x=l$

$$u_2(x,t)|_{x=l}=V_2(x,t)|_{x=l}+A_2(t)+lB_2(t)=f_{2l}(t). \tag{28}$$

From (28) we obtain

$$B_2(t)=\frac{1}{l}f_{2l}(t). \tag{29}$$

The initial conditions for equation (25) take the form

$$u_2(x,t)|_{t=0}=V_2(x,t)|_{t=0}+A_2(t)|_{t=0}+xB_2(t)|_{t=0}\equiv u_2(x). \tag{30}$$

From (30) we obtain

$$V_2(x,t)|_{t=0}=u_2(x)-A_2(t)|_{t=0}-xB_2(t)|_{t=0}\equiv V_{22}(x). \tag{31}$$

Differentiating (30) by t, we obtain

$$
\begin{aligned}
\left.\frac{\partial u_2(x,t)}{\partial t}\right|_{t=0}&=\left.\frac{\partial V_2(x,t)}{\partial t}\right|_{t=0}+\left.\frac{dA_2(t)}{\partial t}\right|_{t=0}+\\
&+x\left.\frac{dB_2(t)}{\partial t}\right|_{t=0}\equiv du_2(x).
\end{aligned} \tag{32}
$$

From (32) we obtain

$$
\begin{aligned}
\left.\frac{\partial V_2(x,t)}{\partial t}\right|_{t=0}&=du_2(x)-\left.\frac{dA_2(t)}{\partial t}\right|_{t=0}-x\left.\frac{dB_2(t)}{\partial t}\right|_{t=0}=\\
&=-\left.\frac{dA_2(t)}{\partial t}\right|_{t=0}-x\left.\frac{dB_2(t)}{\partial t}\right|_{t=0}\equiv V_{22}(x).
\end{aligned} \tag{33}
$$

The equation for the variable $V_1(x, t)$is obtained by substituting (16) in (11), namely

$$\frac{\partial^2 V_1(x,t)}{\partial x^2} = a_{11}(V_1(x,t) + A_1(t) + xB_1(t)) + b_{11}\left(\frac{\partial V_1(x,t)}{dt} + \frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) - c_{11}\left(\frac{\partial V_2(x,t)}{\partial t} + \frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) - d_{11}\frac{\partial^3 V_1(x,t)}{\partial x^2 \partial t} + a_{12}\left(\frac{\partial V_1(x,t)}{\partial t} + \frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{\partial t}\right) + b_{12}\left(\frac{\partial^2 V_1(x,t)}{\partial t^2} + \frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - c_{12}\left(\frac{\partial^2 V_2(x,t)}{\partial t^2} + \frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - d_{12}\frac{\partial^4 V_1(x,t)}{\partial x^2 \partial t^2} + a_{13}\left(\frac{\partial V_2(x,t)}{\partial t} + \frac{dA_2(t)}{dt} + x\frac{dB_1(t)}{\partial t}\right) + b_{13}\left(\frac{\partial^2 V_2(x,t)}{\partial t^2} + \frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - c_{13}\left(\frac{\partial^2 V_1(x,t)}{\partial t^2} + \frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right) - d_{13}\frac{\partial^4 V_2(x,t)}{\partial x^2 \partial t^2}. \quad (34)$$

The equation for the variable $V_2(x, t)$ is obtained by substituting (25) in (12), namely

$$\frac{\partial^2 V_2(x,t)}{\partial x^2} = a_{21}(V_2(x,t) + A_2(t) + xB_2(t)) + b_{21}\left(\frac{\partial V_2(x,t)}{dt} + \frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) - c_{21}\left(\frac{\partial V_1(x,t)}{\partial t} + \frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) - d_{21}\frac{\partial^3 V_2(x,t)}{\partial x^2 \partial t} + a_{22}\left(\frac{\partial V_2(x,t)}{\partial t} + \frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) + b_{22}\left(\frac{\partial^2 V_2(x,t)}{\partial t^2} + \frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - c_{22}\left(\frac{\partial^2 V_1(x,t)}{\partial t^2} + \frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right) - d_{22}\frac{\partial^4 V_2(x,t)}{\partial x^2 \partial t^2} + a_{23}\left(\frac{\partial V_1(x,t)}{\partial t} + \frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) + b_{23}\left(\frac{\partial^2 V_1(x,t)}{\partial t^2} + \frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right) - c_{23}\left(\frac{\partial^2 V_2(x,t)}{\partial t^2} + \frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - d_{23}\frac{\partial^4 V_1(x,t)}{\partial x^2 \partial t^2}. \quad (35)$$

In equations (34, 35) leave the known parts on the right, namely

$$F_1(x,t) = a_{11}(A_1(t) + xB_1(t)) + b_{11}\left(\frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) - c_{11}\left(\frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) + a_{12}\left(\frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) + b_{12}\left(\frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right) - c_{12}\left(\frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) + a_{13}\left(\frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) + b_{13}\left(\frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - c_{13}\left(\frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right); \quad (36)$$

$$F_2(x,t) = a_{21}(A_2(t) + xB_2(t)) + b_{21}\left(\frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) - c_{21}\left(\frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) + a_{22}\left(\frac{dA_2(t)}{dt} + x\frac{dB_2(t)}{dt}\right) + b_{22}\left(\frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right) - c_{22}\left(\frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right) + a_{23}\left(\frac{dA_1(t)}{dt} + x\frac{dB_1(t)}{dt}\right) + b_{23}\left(\frac{d^2 A_1(t)}{dt^2} + x\frac{d^2 B_1(t)}{dt^2}\right) - c_{23}\left(\frac{d^2 A_2(t)}{dt^2} + x\frac{d^2 B_2(t)}{dt^2}\right). \quad (37)$$

Equations for variables $V_1(x, t)$ and $V_2(x, t)$ are written as follows

$$-\frac{\partial^2 V_1(x,t)}{\partial x^2} + a_{11}V_1(x,t) + b_{11}\frac{\partial V_1(x,t)}{\partial t} - c_{11}\frac{\partial V_2(x,t)}{\partial t} - d_{11}\frac{\partial^3 V_1(x,t)}{\partial x^2 \partial t} + a_{12}\frac{\partial V_1(x,t)}{\partial t} + b_{12}\frac{\partial^2 V_1(x,t)}{\partial t^2} - c_{12}\frac{\partial^2 V_2(x,t)}{\partial t^2} - d_{12}\frac{\partial^4 V_1(x,t)}{\partial x^2 \partial t^2} + a_{13}\frac{\partial V_2(x,t)}{\partial t} + b_{13}\frac{\partial^2 V_2(x,t)}{\partial t^2} - c_{13}\frac{\partial^2 V_1(x,t)}{\partial t^2} - d_{13}\frac{\partial^4 V_2(x,t)}{\partial x^2 \partial t^2} = F_1(x,t); \quad (38)$$

$$-\frac{\partial^2 V_2(x,t)}{\partial x^2} + a_{21}V_2(x,t) + b_{21}\frac{\partial V_2(x,t)}{\partial t} - c_{21}\frac{\partial V_1(x,t)}{\partial t} - d_{21}\frac{\partial^3 V_2(x,t)}{\partial^2 x \partial t} + a_{22}\frac{\partial V_2(x,t)}{\partial t} + b_{22}\frac{\partial^2 V_2(x,t)}{\partial t^2} - c_{22}\frac{\partial^2 V_1(x,t)}{\partial t^2} - d_{22}\frac{\partial^4 V_2(x,t)}{\partial x^2 \partial t^2} + a_{23}\frac{\partial V_1(x,t)}{\partial t} + b_{23}\frac{\partial^2 V_1(x,t)}{\partial t^2} - c_{23}\frac{\partial^2 V_2(x,t)}{\partial t^2} - d_{23}\frac{\partial^4 V_1(x,t)}{\partial x^2 \partial t^2} = F_2(x,t). \quad (39)$$

Considering that $V_1(x, t)|_{x=0} = 0$, $V_1(x, t)|_{x=l} = 0$ and $V_2(x, t)|_{x=0} = 0$, $V_2(x, t)|_{x=l} = 0$, we seek solutions for $V_1(x, t)$ and $V_2(x, t)$ as

$$V_1(x,t) = \sum_{k=1}^{m} C_k(t)\sin((\pi kx)/l), \quad 0 < x < l; \quad (40)$$

$$V_2(x,t) = \sum_{k=1}^{m} D_k(t)\sin((\pi kx)/l), \quad 0 < x < l. \quad (41)$$

Find derivatives of (40, 41)

$$\partial V_1(x,t)/\partial t = \sum_{k=1}^{m} dC_k(t)/dt \sin(\pi kx/l); \quad (42)$$

$$\partial V_2(x,t)/\partial t = \sum_{k=1}^{m} dD_k(t)/dt \sin(\pi kx/l). \quad (43)$$

With $t = 0$we obtain

$$V_{1.1}(x,t)\big|_{t=0} = V_{0.1.1}(x) = \sum_{k=1}^{m} C_k(t)\big|_{t=0}\sin(\pi kx/l/); \quad \partial V_{1.2}(x,t)/dt\big|_{t=0} = V_{0.1.2}(x) = \sum_{k=1}^{m} dC(t)/dt\big|_{t=0}\sin(\pi kx/l); \quad (44)$$

$$V_{2.1}(x,t)\big|_{t=0} = V_{0.2.1}(x) = \sum_{k=1}^{m} D_k(t)\big|_{t=0}\sin(\pi kx/l); \quad \partial V_{2.2}(x,t)/dt\big|_{t=0} = V_{0.2.2}(x) = \sum_{k=1}^{m} dD(t)/dt\big|_{t=0}\sin(\pi kx/l). \quad (45)$$

Let us decompose the initial conditions into Fourier series

$$V_{0.1.1}(x) = \sum_{k=1}^{m} \alpha_{k1}\sin(\pi kx/l); \quad V_{0.1.2}(x) = \sum_{k=1}^{m} \beta_{k1}\sin(\pi kx/l); \quad (46)$$

$$V_{0.2.1}(x) = \sum_{k=1}^{m} \alpha_{k2}\sin(\pi kx/l); \quad V_{0.2.2}(x) = \sum_{k=1}^{m} \beta_{k2}\sin(\pi kx/l). \quad (47)$$

From (48, 49) we find

$$\alpha_{k1} = 1/l \int_{x=0}^{l} V_{0.1.1}(x)\sin(\pi kx/l)dx; \quad \beta_{k1} = 1/l \int_{x=0}^{l} V_{0.1.2}(x)\sin(\pi kx/l)dx; \quad (48)$$

$$\alpha_{k2} = 1/l \int_{x=0}^{l} V_{0.2.1}(x)\sin(\pi kx/l)dx;$$
$$\beta_{k2} = 1/l \int_{x=0}^{l} V_{0.2.2}(x)\sin(\pi kx/l)dx. \quad (49)$$

Then from (44, 45) we obtain

$$C_k(t)|_{t=0} = \alpha_{k1} \quad \text{and} \quad dC(t)/dt|_{t=0} = \beta_{k1}; \quad (50)$$

$$D_k(t)|_{t=0} = \alpha_{k2} \quad \text{and} \quad dD(t)/dt|_{t=0} = \beta_{k2}. \quad (51)$$

Differential equations for $C_k(t)$ and $D_k(t)$ were found by decomposing $F_1(x, t)$ and $F_2(x, t)$ into Fourier series, namely

$$F_1(x,t) = \sum_{k=1}^{m} \gamma_k(t)\sin(\pi kx/l); \quad (52)$$

$$F_2(x,t) = \sum_{k=1}^{m} \mu_k(t)\sin(\pi kx/l). \quad (53)$$

Using (40), we transform the left part (38) as follows

$$-d_{12}\partial^4 V_1(x,t)/(\partial x^2 \partial t^2) + b_{12}\partial^2 V_1(x,t)/\partial t^2 - c_{13}\partial^2 V_1(x,t)/\partial t^2 = \sum_{k=1}^{m}(-d_{12}(\pi k/l)^2 + b_{12} + c_{13})d^2C_k(t)/dt^2 \sin(\pi kx/l);$$
$$-d_{11}\partial^3 V_1(x,t)/(\partial x^2 \partial t) + b_{11}\partial V_1(x,t)/\partial t + a_{12}\partial V_1(x,t)/\partial t = \sum_{k=1}^{m}(-d_{11}(\pi k/l)^2 + b_{11} + a_{12})dC_k(t)/dt \sin(\pi kx/l);$$
$$-\partial^2 V_1(x,t)/\partial x^2 + a_{11}\partial V_1(x,t)/\partial t = \sum_{k=1}^{m}(-(\pi k/l)^2 + a_{11})dC_k(t)/dt \sin(\pi kx/l); \quad (54)$$
$$-d_{13}\partial^4 V_2(x,t)/(\partial x^2 \partial t^2) + b_{13}\partial^2 V_2(x,t)/\partial t^2 - c_{12}\partial^2 V_2(x,t)/\partial t^2 = \sum_{k=1}^{m}(-d_{13}(\pi k/l)^2 + b_{13} - c_{12})d^2D_k(t)/dt^2 \sin(\pi kx/l);$$
$$-c_{11}\partial V_2(x,t)/\partial t + a_{13}\partial V_2(x,t)/\partial t = \sum_{k=1}^{m}(-c_{11}(\pi k/l)^2 + a_{13})dD_k(t)/dt \sin(\pi kx/l).$$

Using (41), we transform the left part (39) as follows

$$-d_{22}\partial^4 V_2(x,t)/(\partial x^2 \partial t^2) + b_{22}\partial^2 V_2(x,t)/\partial t^2 - c_{23}\partial^2 V_2(x,t)/\partial t^2 = \sum_{k=1}^{m}(-d_{22}(\pi k/l)^2 + b_{22} - c_{23})d^2D_k(t)/dt^2 \sin(\pi kx/l);$$
$$-d_{21}\partial^3 V_2(x,t)/(\partial x^2 \partial t) + b_{21}\partial V_2(x,t)/\partial t + a_{22}\partial V_2(x,t)/\partial t = \sum_{k=1}^{m}(-d_{21}(\pi k/l)^2 + b_{21} + a_{22})dD_k(t)/dt \sin(\pi kx/l); \quad (55)$$
$$-\partial^2 V_2(x,t)/\partial x^2 + a_{21}\partial V_2(x,t)/\partial t = \sum_{k=1}^{m}(-(\pi k/l)^2 + a_{21})dD_k(t)/dt \sin(\pi kx/l);$$
$$-d_{23}\partial^4 V_1(x,t)/(\partial x^2 \partial t^2) + b_{23}\partial^2 V_{11}(x,t)/\partial t^2 - c_{22}\partial^2 V_1(x,t)/\partial t^2 = \sum_{k=1}^{m}(-d_{23}(\pi k/l)^2 + b_{23} - c_{22})d^2C_k(t)/dt^2 \sin(\pi kx/l);$$
$$-c_{21}\partial V_1(x,t)/\partial t + a_{23}\partial V_1(x,t)/\partial t = \sum_{k=1}^{m}(-c_{21}(\pi k/l)^2 + a_{23})dC_k(t)/dt \sin(\pi kx/l).$$

According to (38, 40) for (52) we enter the notation

$$a_1 = -d_{12}\left(\frac{\pi k}{l}\right)^2 + b_{12} - c_{13};$$
$$b_1 = -d_{11}\left(\frac{\pi k}{l}\right)^2 + b_{11} + a_{12}; \quad c_1 = a_{11} - \left(\frac{\pi k}{l}\right)^2; \quad (56)$$
$$g_{12} = -d_{13}\left(\frac{\pi k}{l}\right)^2 + b_{13} - c_{12}; \quad k_{12} = -c_{11} + a_{13}.$$

According to (39, 41) for (55) enter the notation

$$a_2 = -d_{22}\left(\frac{\pi k}{l}\right)^2 + b_{22} - c_{23};$$
$$b_2 = -d_{21}\left(\frac{\pi k}{l}\right)^2 + b_{21} + a_{22}; \quad c_2 = a_{21} - \left(\frac{\pi k}{l}\right)^2; \quad (57)$$
$$g_{21} = -d_{23}\left(\frac{\pi k}{l}\right)^2 + b_{23} - c_{22}; \quad k_{21} = -c_{21} + a_{23}.$$

Taking into account (38, 40) and entering the notation (56), obtain

$$a_1\frac{d^2C_k(t)}{dt^2} + b_1\frac{dC_k(t)}{dt} + c_1C_k(t) + g_{12}\frac{d^2D_k(t)}{dt^2} + k_{12}\frac{dD_k(t)}{dt} = \gamma_k(t), \quad k = \overline{1,m}. \quad (58)$$

Taking into account (39, 41) and entering the notation (57), obtain

$$a_2\frac{d^2D_k(t)}{dt^2} + b_2\frac{dD_k(t)}{dt} + c_2D_k(t) + g_{21}\frac{d^2C_k(t)}{dt^2} + k_{21}\frac{dC_k(t)}{dt} = \mu_k(t), \quad k = \overline{1,m}. \quad (59)$$

Consider equations (58, 59) as uniform with constant coefficients and zero initial conditions

$$a_1\frac{d^2e_1(t)}{dt^2} + b_1\frac{de_1(t)}{dt} + c_1e_1(t) + g_{12}\frac{d^2e_2(t)}{dt^2} + k_{12}\frac{de_2(t)}{dt} = 0; \quad (60)$$

$$a_2\frac{d^2e_2(t)}{dt^2} + b_2\frac{de_2(t)}{dt} + c_2e_2(t) + g_{21}\frac{d^2e_1(t)}{dt^2} + k_{21}\frac{de_1(t)}{dt} = 0. \quad (61)$$

Equations (60, 61) are written in operator form

$$(a_1\lambda^2 + b_1\lambda + c_1)e_1(\lambda) + (g_{12}\lambda^2 + k_{12}\lambda)e_2(\lambda) = 0; \quad (62)$$

$$(a_2\lambda^2 + b_2\lambda + c_2)e_2(\lambda) + (g_{21}\lambda^2 + k_{21}\lambda)e_1(\lambda) = 0. \quad (63)$$

The characteristic equation of the system of equations (62, 63) has the form

$$d_4\lambda^4 + d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0 = 0, \quad (64)$$

where $d_4 = a_1a_2 + g_{21}g_{12}$; $d_3 = a_1b_2 + a_2b_1 - g_{21}k_{12} - g_{12}k_{21}$; $d_2 = a_1c_2 + b_2b_1 + a_2c_1 - k_{21}k_{12}$; $d_1 = b_1c_2 + b_2c_1$; $d_0 = c_1c_2$.

Consider the case when the characteristic equation (64) has complex conjugate roots $\lambda_{1,2} = \delta_1 \pm j\theta_1$ and $\lambda_{3,4} = \delta_2 \pm j\theta_2$.

Then uniform equations (62, 63) have the following solution $e_1(t) = e^{\delta_1 t}\cos(\theta_1 t)$; $e_2(t) = e^{\delta_1 t}\sin(\theta_1 t)$ and $e_3(t) = e^{\delta_2 t}\cos(\theta_2 t)$; $e_4(t) = e^{\delta_2 t}\sin(\theta_2 t)$ accordingly.

By the method of variation of arbitrary constants we look for solution of equations (58, 59) in the form [1]

$$C_k(t) = B_1(t)e_1(t) + B_2(t)e_2(t) + B_3(t)e_3(t) + B_4(t)e_4(t); \quad (65)$$

$$D_k(t) = B_3(t)e_3(t) + B_4(t)e_4(t) + B_1(t)e_1(t) + B_2(t)e_2(t). \quad (66)$$

Functions $B_1(t)$, $B_2(t)$, $B_3(t)$, $B_4(t)$, are found from the system of equations

$$\begin{aligned} &dB_1(t)/dte_1(t) + dB_2(t)/dte_2(t) + dB_3(t)/dte_3(t) + \\ &+ dB_4(t)/dte_4(t) = 0; \\ &dB_1(t)/dtde_1(t)/dt + dB_2(t)/dtde_2(t)/dt + \\ &+ dB_3(t)/dtde_3(t)/dt + dB_4(t)/dtde_4(t)/dt = 1/a_1\gamma_k(t); \\ &dB_3(t)/dte_3(t) + dB_4(t)/dte_4(t) + dB_1(t)/dte_1(t) + \\ &+ dB_2(t)/dte_2(t) = 0; \\ &dB_3(t)/dtde_3(t)/dt + dB_4(t)/dtde_4(t)/dt + \\ &+ dB_1(t)/dtde_1(t)/dt + dB_2(t)/dtde_2(t)/dt = 1/a_2\mu_k(t). \end{aligned} \quad (67)$$

Entering the notation

$$\Delta = \begin{vmatrix} e_1(t) & e_2(t) & e_3(t) & e_4(t) \\ \dfrac{de_1(t)}{dt} & \dfrac{de_2(t)}{dt} & \dfrac{de_3(t)}{dt} & \dfrac{de_4(t)}{dt} \\ e_3(t) & e_4(t) & e_1(t) & e_2(t) \\ \dfrac{de_3(t)}{dt} & \dfrac{de_4(t)}{dt} & \dfrac{de_1(t)}{dt} & \dfrac{de_2(t)}{dt} \end{vmatrix}. \quad (68)$$

We also find Δ_1, Δ_2, Δ_3 and Δ_4 accordingly replacing the columns in (68) with the right part (67); find

$$\begin{aligned} &dB_1(t)/dt = \Delta_1/\Delta; \quad dB_2(t)/dt = \Delta_2/\Delta; \\ &dB_3(t)/dt = \Delta_3/\Delta; \quad dB_4(t)/dt = \Delta_4/\Delta. \end{aligned} \quad (69)$$

To find the initial conditions $B_1(t)|_{t=0}$; $B_2(t)|_{t=0}$; $B_3(t)|_{t=0}$; $B_4(t)|_{t=0}$, we differentiate (65, 66), and using the first and third equations from (67), obtain

$$\begin{aligned} &dC_k(t)/dt = dB_1(t)/dte_1(t) + dB_2(t)/dte_2(t) + \\ &+ dB_3(t)/dte_3(t) + dB_4(t)/dte_4(t) + B_1(t)de_1(t)/dt + \\ &+ B_2(t)de_2(t)/dt + B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt = \\ &= B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt + B_3(t)de_3(t)/dt + \\ &+ B_4(t)de_4(t)/dt; \end{aligned} \quad (70)$$

$$\begin{aligned} &dD_k(t)/dt = dB_3(t)/dte_3(t) + dB_4(t)/dte_4(t) + \\ &+ dB_1(t)/dte_1(t) + dB_2(t)/dte_2(t) + B_3(t)de_3(t)/dt + \\ &+ B_4(t)de_4(t)/dt + B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt = \\ &= B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_1(t)de_1(t)/dt + \\ &+ B_2(t)de_2(t)/dt. \end{aligned} \quad (71)$$

From equations (70, 71), as well as equations (65, 66) by $t = 0$ obtain a system of equations

$$\begin{aligned} &C_k(t)\big|_{t=0} = B_1(t)\big|_{t=0} e_1(t)\big|_{t=0} + B_2(t)\big|_{t=0} e_2(t)\big|_{t=0} + \\ &+ B_3(t)\big|_{t=0} e_3(t)\big|_{t=0} + B_4(t)\big|_{t=0} e_4(t)\big|_{t=0}; \\ &dC_k(t)/dt\big|_{t=0} = B_1(t)\big|_{t=0} de_1(t)/dt\big|_{t=0} + B_2(t)\big|_{t=0} de_2(t)/dt\big|_{t=0} + \\ &+ B_3(t)\big|_{t=0} de_3(t)/dt\big|_{t=0} + B_4(t)\big|_{t=0} de_4(t)/dt\big|_{t=0}; \\ &D_k(t)\big|_{t=0} = B_3(t)\big|_{t=0} e_3(t)\big|_{t=0} + B_4(t)\big|_{t=0} e_4(t)\big|_{t=0} + \\ &+ B_1(t)\big|_{t=0} e_1(t)\big|_{t=0} + B_2(t)\big|_{t=0} e_2(t)\big|_{t=0}; \\ &dD_k(t)/dt\big|_{t=0} = B_3(t)\big|_{t=0} de_3(t)/dt\big|_{t=0} + B_4(t)\big|_{t=0} de_4(t)/dt\big|_{t=0} + \\ &+ B_1(t)\big|_{t=0} de_1(t)/dt\big|_{t=0} + B_2(t)\big|_{t=0} de_2(t)/dt\big|_{t=0}. \end{aligned} \quad (72)$$

Given (50, 51), find $B_1(t)|_{t=0}$; $B_2(t)|_{t=0}$; $B_3(t)|_{t=0}$; $B_4(t)|_{t=0}$ from (72) and according to (68) we find

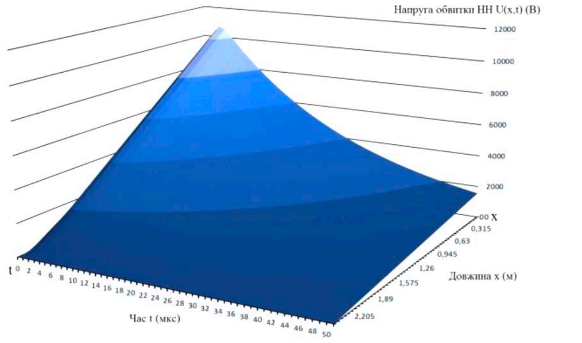

Fig. 1. Dependence of voltage on distance and time. High voltage winding of the transformer

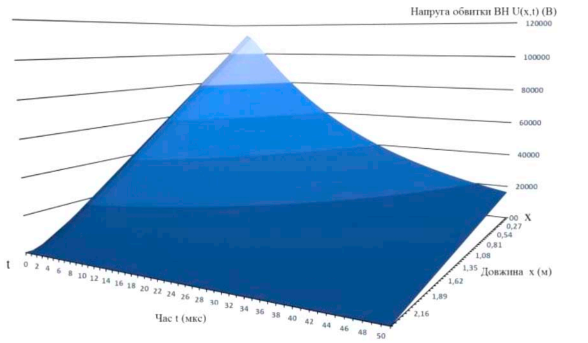

Fig. 2. Dependence of voltage on distance and time. Low voltage winding of the transformer

$$\begin{aligned} &B_1(t) = \int_0^t dB_1(t)/dt + B_1(t)\big|_{t=0}; \\ &B_2(t) = \int_0^t dB_2(t)/dt + B_2(t)\big|_{t=0}; \\ &B_3(t) = \int_0^t dB_3(t)/dt + B_3(t)\big|_{t=0}; \\ &B_4(t) = \int_0^t dB_4(t)/dt + B_4(t)\big|_{t=0}. \end{aligned} \quad (73)$$

According to (65, 66), find $C_k(t)$ and $D_k(t)$, and according to (40, 41), we find $V_1(x, t)$ and $V_2(x, t)$ and for (16, 25) we determine $u_1(x, t)$ and $u_2(x, t)$, namely, the solution of equations (11, 12).

Figs. 1, 2 show the voltage changes in the primary and secondary windings of the transformer under the action of the voltage pulse at the input of the primary winding depending on the distance and time.

Conclusion. The improved mathematical model is created, which allows analyzing the wave processes in power transformers, with adequate consideration of electromagnetic connections between windings, initial and boundary conditions. In turn, this will allow calculating the transient processes, creating surge protection and coordinating their isolation. A modified Fourier method, which allows solving a system of partial differential equations of wave processes in transformers, is proposed.

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Математичне моделювання хвильових процесів у двообвиткових трансформаторах з урахуванням основного магнітного потоку

М. С. Сегеда, Є. В. Черемних, П. Ф. Гоголюк, Ю. В. Близнак

Національний університет «Львівська політехніка», м. Львів, Україна, e-mail: mykhailo.s.seheda@lpnu.ua

Мета. Розроблення методу математичного моделювання хвильових процесів у силових двообвиткових трансформаторах на підставі заступної схеми, що враховує конструктивні особливості силових трансформаторів.

Методика. Формування математичних моделей для дослідження хвильових процесів у силових двообвиткових трансформаторах, а також подальший розвиток аналітичного методу розв'язання системи диференційних рівнянь із частинними похідними.

Результати. Створена математична модель для дослідження хвильових процесів у силових двообвиткових трансформаторах на підставі заступної схеми, що адекватно враховує як електричні, так і магнетні зв'язки, а також запропоновано удосконалений аналітичний метод розв'язання системи диференційних рівнянь із частинними похідними, що дозволяє враховувати співвідношення між інтервалом часу поширення елетромагнетних хвиль уздовж усієї довжини обвиток та інтервалом часу, упродовж якого напруга змінюється істотніше від повної її зміни під час хвильових процесів.

Наукова новизна. У роботі запропонована математична модель для дослідження хвильових процесів у обвитках силових двообвиткових трансформаторах на підставі його заступної схеми, що враховує електричні й магнетні зв'язки, а також удосконалено метод Фур'є для розв'язання системи диференційних рівнянь із частинними похідними.

Практична значимість. Ураховуючи, що до роботи силових трансформаторів ставляться високі вимоги стосовно надійності їх роботи, створена математична модель для розрахунку хвильових процесів у обвитках силових трансформаторів, яка дозволяє здійснювати аналіз розподілу напруги вздовж обвиток трансформатора під час дії на них імпульсної напруги, що дасть змогу коригувати їхню ізоляційну спроможність.

Ключові слова: *хвильовий процес, математична модель, трансформатор, диференційні рівняння з частинними похідними, крайова задача*

Recommended for publication by Yu. I. Tugay, Doctor of Technical Sciences. The manuscript was submitted 16.03.21.