UDC 614.841.34

https://doi.org/10.33271/nvngu/2020-3/027

R. M. Tatsiy, Dr. Sc. (Phys.-Math.), Prof.,
orcid.org/0000-0001-7764-2528,
O. Yu. Pazen, Cand. Sc. (Tech.),
orcid.org/0000-0003-1655-3825,
S. Ya. Vovk, Cand. Sc. (Tech.),
orcid.org/0000-0001-7007-7263,
D. V. Kharyshyn, Cand. Sc. (Tech.),
orcid.org/0000-0002-0927-9998

Lviv State University of Life Safety, Lviv, Ukraine, e-mail: opazen@gmail.com

SIMULATION OF HEAT TRANSFER PROCESS IN A MULTILATERAL CYLINDRICAL SHELL TAKING INTO ACCOUNT THE INTERNAL HEAT SOURCES

Purpose. To investigate the peculiarities of distribution of a non-stationary temperature field over the thickness of a multilayer hollow cylinder under convective heat exchange conditions on its surfaces, taking into account the presence of internal (distributed) heat sources.

Methodology. In order to achieve this goal, a direct method of solving boundary value problems of the theory of thermal conductivity was applied, which includes the application of the method of reduction, the concept of quasi derivatives, the method of separation of variables, and the modified method of Fourier eigenfunctions.

Findings. The solution of the boundary value problem was obtained in a closed form, which allowed us to create an algorithm for calculating the propagation of a non-stationary temperature field in multilayer hollow cylindrical structures under convective heat exchange on its surfaces and the presence of internal heat sources. It should be noted that such algorithms include only: a) finding the roots of the characteristic equation; b) multiplication of finite number of known (2×2) matrices; c) calculation of defined integrals; d) summing the required number of members of the series to obtain the specified accuracy. Changing the third-order boundary conditions to any other boundary conditions does not cause any significant difficulty in solving the problem.

Originality. A closed solution is obtained for the propagation of a non-stationary temperature field in a multilayer hollow cylinder in the presence of internal sources of heat and convective heat exchange on its surfaces.

Practical value. Implementation of the research results allows us to investigate the processes of heating or cooling multilayer hollow structures, taking into account the internal heat sources encountered in several applied problems. These are tasks that can be related to the processes of cooling of thermal elements of nuclear power plants, changes in the temperature field during microarray oxidation, heating of electronic components during the passage of electric current.

Keywords: *thermal conductivity, non-stationary temperature field, hollow cylinder*

Introduction. Research studies on heat exchange processes in multilayer cylindrical structures, taking into account the presence of internal distributed heat sources, do not lose their relevance. Such tasks are widely used, as they are increasingly encountered in various industries: construction, (the process of evaporation of moisture when heated hollow concrete columns), oil and gas industry (cylindrical tanks, oil and gas pipelines), aerospace and energy industry (cylindrical elements in reactors of nuclear power plants) and in other various fields of engineering as structural elements and machine parts. So, for example, for oil and gas engineering, one of the modern tasks is to increase the reliability and durability of machine parts by strengthening the working surfaces with micro-arc oxidation, during which internal heat sources arise [1, 2]. In the electrical engineering field, such problems arise when electrical current passes through electronic elements of cylindrical shape (capacitors, resistors, and others).

The main characteristic feature of such multilayer elements is the combination of various mechanical and thermophysical characteristics of the layers, which makes them more sophisticated. However, this approach causes considerable difficulties in the development of analytical methods for their research. Therefore, the development of new methods for the research of multilayer, in particular, cylindrical structures is a relevant problem of today.

Literature review. Numerous publications are devoted to solving the problem of heat transfer. The basic methods for researching the problems of determining the distribution of a non-stationary temperature field in multilayered structures are conditionally divided into three types: a) direct or classical ones, based on the method of separation of variables [3]; b) Laplace transform operation, using various kinds of integral transformations [4]; c) approximate analytical and numerical methods [5]. Thus, in [6] the distribution of the temperature field in a multilayer hollow cylindrical structure with the inclusion of an internal heat source by the Laplace integral transformation method is investigated. In the [7] a method is proposed for solving the problem of thermal conductivity for an arbitrary number of layers, but without considering heat sources.

In recent years, multilayer hollow cylindrical structures are considered in [8, 9]. The basis of these publications is a direct (classical) scheme of research based on the method of reduction, the concept of quasi-derivatives, a modern theory of systems of linear differential equations, modified method of Fourier eigenfunctions.

Unsolved aspects of the problem. Theoretically, analytical methods should be applied to multilayered structures. However, in practice, the number of layers is usually limited to two or three [7]. This is conditioned by the fact that the increasing number of layers leads to cumbersome calculations. Therefore, the problem of efficient analytical solution of the problem of thermal conductivity in multilayer hollow cylindrical structures, taking into account the presence of internal heat sources, remains relevant.

Problem statement and its mathematical model. Let us consider a multilayered cylindrical structure whose area is bounded by radii $r = r_0$ and $r = r_n$ and is divided into n layers. Each layer is made of isotropic material and is endowed with its own coefficient of thermal conductivity λ, W/m · °C, specific heat capacity c, J/kg · °C, and density ρ, kg/m^3. In addition, the layers of the structure provide for the presence of internal heat sources q_v, W/m^3 [10], while the temperature t, °C, depends on the coordinate r, m, and time τ, sec.

This formulation of the problem is reduced to solving the differential equation of thermal conductivity [8]

$$c(r)\rho(r)\frac{\partial t(r,\tau)}{\partial \tau}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda(r)\frac{\partial t(r,\tau)}{\partial r}\right)+q_v(r). \qquad (1)$$

We assume that there is convective heat exchange with the environment on the outer and inner surfaces of the cylinder, that is, there are boundary conditions of the third kind [9]

$$\begin{cases}\lambda\frac{\partial t}{\partial r}(r_0,\tau)=\alpha_0\left(t(r_0,\tau)-\psi_0(\tau)\right)\\ -\lambda\frac{\partial t}{\partial r}(r_n,\tau)=\alpha_n\left(t(r_n,\tau)-\psi_n(\tau)\right)\end{cases}. \qquad (2)$$

Also known at the initial time is the distribution of the temperature field (initial condition)

$$t(r,0)=\varphi(r). \qquad (3)$$

Further, we will use the following notation [8]: θ_i – characteristic function of the half-open interval $[r_i, r_{i+1})$, so, $\theta_i = \begin{cases}1, r\in[r_i,r_{i+1})\\ 0, r\notin[r_i,r_{i+1})\end{cases}$; $t^{[1]}(r,\tau)\overset{df}{=}r\lambda\frac{\partial t}{\partial r}$ – quasi-derivative, $q(r,\tau)=\frac{t^{[1]}(r,\tau)}{r}$ – density of heat flow. Thermophysical characteristics of materials will be considered constant values for each of the layers, so we will represent them as piecewise constant functions $\lambda(r)=\sum_{i=0}^{n-1}\lambda_i\theta_i$; $c(r)\cdot\rho(r)=\sum_{i=0}^{n-1}c_i\cdot\rho_i\cdot\theta_i$; $q_v(r)=\sum_{i=0}^{n-1}q_{vi}\cdot\theta_i$; $\varphi(r)=\sum_{i=0}^{n-1}\varphi_i\cdot\theta_i$; $\lambda_i, c_i, \rho_i, q_{vi}\in R$; $\lambda_i, c_i, \rho_i>0$; $\forall i=\overline{0,n-1}$.

By entering the notation of the quasi-derivative and multiplying the boundary conditions (2) by r, we obtain

$$\begin{cases}\alpha_0 r_0 t(r_0,\tau)-t^{[1]}(r_0,\tau)=\alpha_0 r_0\psi_0(\tau)\\ \alpha_n r_n t(r_n,\tau)+t^{[1]}(r_n,\tau)=\alpha_n r_n\psi_n(\tau)\end{cases}.$$

In mathematical physics, the well-known method of reduction [8], which takes into account the inhomogeneity of boundary conditions and is associated with the separation of the quasi-stationary part. Therefore, the solution of problem (1–3) will be sought as the sum of two related functions

$$t(r,\tau)=u(r,\tau)+v(r,\tau). \qquad (4)$$

Any function $u(r,\tau)$ or $v(r,\tau)$ can be chosen in a special way, and then the other will be uniquely determined.

Function selection $u(r,\tau)$ and mixed problem for $v(r,\tau)$. *Exercise for function $u(r,\tau)$.* We define the function $u(r,\tau)$ as a solution to a quasi-stationary boundary value problem

$$\frac{1}{r}\left(r\lambda u'\right)'+q_v=0; \qquad (5)$$

$$\begin{cases}\alpha_0 r_0 u(r_0,\tau)-u^{[1]}(r_0,\tau)=\alpha_0 r_0\psi_0(\tau)\\ \alpha_n r_n u(r_n,\tau)+u^{[1]}(r_n,\tau)=\alpha_n r_n\psi_n(\tau)\end{cases}, \qquad (6)$$

where $u^{[1]}\overset{df}{=}r\lambda\frac{\partial u(r,\tau)}{\partial r}$ and also later on $v^{[1]}\overset{df}{=}r\lambda\frac{\partial v(r,\tau)}{\partial r}$ – quasi-derivative.

By entering vectors $\mathbf{u}=(u, \quad u^{[1]})^T$, $\mathbf{q}=(0 \quad rq_v)^T$ and matrix $\mathbf{A}=\begin{pmatrix}0 & \frac{1}{r\lambda}\\ 0 & 0\end{pmatrix}$ we reduce the differential equation (5) to the equivalent system of first order differential equations

$$\mathbf{u}'=\mathbf{A}\mathbf{u}-\mathbf{q}. \qquad (7)$$

We also write the boundary conditions (6) in vector form

$$\mathbf{P}\cdot\mathbf{u}(r_0,\tau)+\mathbf{Q}\cdot\mathbf{u}(r_n,\tau)=\Gamma(\tau), \qquad (8)$$

where $\mathbf{P}$, $\mathbf{Q}$ and $\Gamma(\tau)$ have the appearance

$$P=\begin{pmatrix}\alpha_0 r_0 & -1\\ 0 & 0\end{pmatrix};\quad Q=\begin{pmatrix}0 & 0\\ \alpha_n r_n & 1\end{pmatrix};\quad \Gamma(\tau)=\begin{pmatrix}\alpha_0 r_0\psi_0(\tau)\\ \alpha_n r_n\psi_n(\tau)\end{pmatrix}.$$

By the solution of system (7) we mean absolutely continuous in the interval $[r_0, r_n]$ vector function $\mathbf{u}(r)$ which justifies this system almost everywhere except possibly the break points of the coefficients c, ρ, λ, q_v.

At each interval $[r_i, r_{i+1})$ system (7) has the appearance

$$\mathbf{u}_i'=\mathbf{A}_i\mathbf{u}_i-\mathbf{q}_i;\quad \mathbf{A}_i=\begin{pmatrix}0 & \frac{1}{r\lambda_i}\\ 0 & 0\end{pmatrix};\quad \mathbf{q}_i=\begin{pmatrix}0\\ rq_{vi}\end{pmatrix}. \qquad (9)$$

For the corresponding (9) homogeneous system $\mathbf{u}_i'=\mathbf{A}_i\mathbf{u}_i$ we will consider the known Cauchy matrix $\mathbf{B}_i(r,s)$ which has the following properties:

1) by the variable r, it fulfills the matrix equation

$$\frac{\partial\mathbf{B}_i(r,s)}{\partial r}=\mathbf{A}_i\mathbf{B}_i(r,s);$$

2) $\mathbf{B}_i(s,s)=\mathbf{E}$, where $\mathbf{E}$ is the unit matrix.

By direct verification we make sure that

$$\mathbf{B}_i(r,s)=\begin{pmatrix}1 & K_i(r,s)\\ 0 & 1\end{pmatrix}, \qquad (10)$$

satisfies the conditions 1) – 2), where $K_i(r,s)=\frac{1}{\lambda_i}\int_s^r\frac{dz}{z}$.

For arbitrary $k\geq i$ we denote

$$\mathbf{B}(r_k,r_i)\overset{df}{=}\mathbf{B}_{k-1}(r_k,r_{k-1})\cdot\mathbf{B}_{k-2}(r_{k-1},r_{k-2})\cdots\mathbf{B}_i(r_{i+1},r_i). \qquad (11)$$

The structure (10) of the matrix $\mathbf{B}_i(r,s)$ allows establishing the structure of the matrix (11), namely

$$\mathbf{B}(r_k,r_m)=\begin{pmatrix}1 & \sum_{i=m}^{k-1}K_i(r,s)\\ 0 & 1\end{pmatrix},$$

however, $\mathbf{B}(r_m,r_m)\overset{df}{=}\mathbf{E}$.

It is established that at each of the intervals $[r_i, r_{i+1})$ the solution of the problem (5, 6) is represented as a vector function $\mathbf{u}_i(r,\tau)$, where the first coordinate is the desired function $u_i(r,\tau)$, as the solution of equation (5) and the second one is its quasi-derivative

$$\mathbf{u}_i(r,\tau)=\mathbf{B}_i(r,r_i)\cdot\mathbf{B}(r_i,r_0)\cdot\mathbf{P}_0+ \\ +\mathbf{B}_i(r,r_i)\cdot\sum_{k=1}^{i}\mathbf{B}(r_i,r_k)\cdot\mathbf{Z}_k+\int_{r_i}^{r}\mathbf{B}_i(r,s)\cdot\mathbf{q}_i(s)ds, \qquad (12)$$

where $\mathbf{P}_0=(\mathbf{P}+\mathbf{Q}\mathbf{B}(r_n,r_0))^{-1}\cdot(\Gamma-\mathbf{Q}\sum_{k=1}^{n}\mathbf{B}(r_n,r_k)\cdot\mathbf{Z}_k)$;

$$\mathbf{B}_i(r,s)=\begin{pmatrix}1 & K_i(r,s)\\ 0 & 1\end{pmatrix};\quad K_i(r,s)=\frac{1}{\lambda_i}\ln\frac{r}{s};$$

$$\mathbf{B}(r_i,r_0)=\begin{pmatrix}1 & K(r_i,r_0)\\ 0 & 1\end{pmatrix};\quad K(r_i,r_0)=\sum_{k=0}^{i-1}\frac{1}{\lambda_k}\ln\frac{r_{k+1}}{r_k};$$

$$\mathbf{Z}_k=-\begin{pmatrix}\frac{q_{v,k-1}}{\lambda_{k-1}}\left[\frac{1}{4}\left(r_k^2-r_{k-1}^2\right)-\frac{r_{k-1}}{2}\ln\frac{r_k}{r_{k-1}}\right]\\ \frac{q_{v,k-1}}{2}\left(r_k^2-r_{k-1}^2\right)\end{pmatrix}=\begin{pmatrix}z_k\\ z_k^{[1]}\end{pmatrix},$$

$$k=\overline{1,n-1}.$$

$$\int_{r_i}^{r} \mathbf{B}_i(r,s)\mathbf{q}_i ds = -\begin{pmatrix} \frac{q_{vi}}{\lambda_i}\left[\frac{1}{4}\left(r^2 - r_i^2\right) - \frac{r_i}{2}\ln\frac{r}{r_i}\right] \\ \frac{q_{vi}}{2}\left(r^2 - r_i^2\right) \end{pmatrix}, \quad i = \overline{0, n-1}.$$

Formula (12) allows writing the solution of problem (5, 6) in the interval $[r_0, r_n]$ using characteristic function θ_i such as

$$\mathbf{u}(r,\tau) = \sum_{i=0}^{n-1} \mathbf{u}_i(r,\tau)\theta_i.$$

Boundary value problem for function $v(r, \tau)$. Applying formula (4) to equation (1) we obtain

$$c\rho\frac{\partial u(r,\tau)}{\partial\tau} + c\rho\frac{\partial v(r,\tau)}{\partial\tau} = \\ = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial u(r,\tau)}{\partial r}\right) + q_v + \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial v(r,\tau)}{\partial r}\right). \quad (13)$$

Because $u(r, \tau)$ is the solution of problem (5–6), then in (13) it should be taken into account that $\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial u(r,\tau)}{\partial r}\right) + q_v \equiv 0.$ Therefore, we arrive at a nonuniform differential equation to define a function $v(r, \tau)$

$$c\rho\frac{\partial v(r,\tau)}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial v(r,\tau)}{\partial r}\right) - c\rho\frac{\partial u(r,\tau)}{\partial\tau}. \quad (14)$$

Regarding the function $-c\rho\frac{\partial u(r,\tau)}{\partial\tau}$ in the right-hand side of (14), we consider it known because the function is known $u(r, \tau)$ as a solution to problem (5, 6). Because the same function $u(r, \tau)$ is true of boundary conditions (6), so from (4) we obtain boundary conditions for the function $v(r, \tau)$

$$\begin{cases} \alpha_0 r_0 v(r_0,\tau) - v^{[1]}(r_0,\tau) = 0 \\ \alpha_n r_n v(r_n,\tau) + v^{[1]}(r_n,\tau) = 0 \end{cases}. \quad (15)$$

The initial condition for $v(r, \tau)$ will look like

$$v(r,0) = f(r) \equiv \varphi(r) - u(r,0) = \sum_{i=0}^{n-1}\left[\varphi_i(r) - u_i(r,0)\right]\theta_i. \quad (16)$$

So, if the solution $u(r, \tau)$ of tasks (5, 6) is known, function $v(r, \tau)$ is the solution to a mixed task (14–16).

Separation of variables and the eigenvalue problems. We will look for non-trivial solutions of homogeneous differential equation

$$c\rho\frac{\partial v(r,\tau)}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial v(r,\tau)}{\partial r}\right), \quad (17)$$

which satisfies the boundary conditions (15) in the form [8]

$$v(r, \tau) = e^{-\omega\tau} \cdot R(r), \quad (18)$$

where ω is the parameter, and $R(r)$ is unknown function.

Substituting the right-hand side (18) into equation (17) and boundary conditions (15) we obtain a quasi-differential equation

$$(r\lambda R')' + \omega c\rho r R = 0, \quad (19)$$

with boundary conditions

$$\begin{cases} \alpha_0 r_0 R(r_0) - R^{[1]}(r_0) = 0 \\ \alpha_n r_n R(r_n) + R^{[1]}(r_n) = 0 \end{cases}, \quad (20)$$

where $R^{[1]} \overset{df}{=} r\lambda R'$ – quasi-derivative.

Boundary value problem (19, 20) is a classical eigenvalue problem where it is necessary to find the parameter value ω (eigenvalues ω_k) in which there are corresponding non-trivial solutions (eigenfunctions) $R_k(r, \omega_k)$. The properties of eigenvalues and eigenfunctions of this problem are studied in detail and described in [8]. Just note that all of the eigenvalues ω_k are positive and different, but eigenfunctions $R_k(r, \omega_k)$ are orthogonal with weight $c\rho r$ [8], that is,

$$\int_{r_0}^{r_n} R_i(r,\omega_i) \cdot R_j(r,\omega_j) c\rho r dr = 0, \ i \neq j.$$

Structural construction of eigenfunctions. By entering a vector $\mathbf{R} = (R, \quad R^{[1]})^T$ and matrix $\widetilde{\mathbf{A}} = \begin{pmatrix} 0 & \frac{1}{r\lambda} \\ -\omega r c\rho & 0 \end{pmatrix}$, we reduce the quasi-differential equation (19) to the equivalent system of first-order differential equations

$$\mathbf{R}' = \widetilde{\mathbf{A}} \cdot \mathbf{R}. \quad (21)$$

Appropriate system at intervals $[r_i, r_{i+1})$ is written in the form

$$\mathbf{R}_i' = \widetilde{\mathbf{A}}_i \cdot \mathbf{R}_i, \quad \widetilde{\mathbf{A}}_i = \begin{pmatrix} 0 & \frac{1}{r\lambda_i} \\ -\omega r c_i \rho_i & 0 \end{pmatrix}, \quad i = \overline{0, n-1}. \quad (22)$$

In work [8] it is established that the Cauchy matrix $\tilde{\mathbf{B}}_i(r,s,\omega)$ system (22) looks like

$$\tilde{B}_i(r,s,\omega) = \begin{pmatrix} b_{11}^i & b_{12}^i \\ b_{21}^i & b_{22}^i \end{pmatrix};$$

$$b_{11}^i = \frac{\pi\beta_i s\left(J_1(\beta_i, s)Y_0(\beta_i, r) - J_0(\beta_i, r)Y_1(\beta_i, s)\right)}{2};$$

$$b_{12}^i = \frac{\pi\left(J_0(\beta_i, s)Y_0(\beta_i, r) - J_0(\beta_i, r)Y_0(\beta_i, s)\right)}{2\lambda_i};$$

$$b_{21}^i = \frac{\pi\lambda_i\beta_i^2 rs\left(J_1(\beta_i, r)Y_1(\beta_i, s) - J_1(\beta_i, s)Y_1(\beta_i, r)\right)}{2};$$

$$b_{22}^i = \frac{\pi\beta_i r\left(J_1(\beta_i, r)Y_0(\beta_i, s) - J_0(\beta_i, s)Y_1(\beta_i, r)\right)}{2}, \quad i = \overline{0, n-1},$$

where $\beta_i = \sqrt{\frac{\omega c_i \rho_i}{\lambda_i}}$, and J_0, J_1 and Y_0, Y_1 are function Bessel and Neumann zero and first order respectively.

We will seek non-trivial solutions $\mathbf{R}(r, \omega)$ of system (21) in the form [8]

$$\mathbf{R}(r,\omega) = \tilde{B}(r, r_0, \omega) \cdot \mathbf{C}, \quad (23)$$

where

$$\tilde{B}(r, r_0, \omega) = \tilde{B}_0(r, r_0, \omega)\theta_0 + \tilde{B}_1(r, r_1, \omega)\tilde{B}_0(r_1, r_0, \omega)\theta_1 + \ldots + \\ + \tilde{B}_{n-1}(r, r_{n-1}, \omega)\prod_{i=1}^{n-1}\tilde{B}_{i-1}(r_i, r_{i-1}, \omega),$$

and $\mathbf{C} = (C_1, \quad C_2)^T$ – some non-zero vector.

Applying to equality (23) boundary conditions (8) (if $\Gamma(\tau) \equiv \mathbf{0}$), we will get

$$P \cdot \mathbf{R}(r_0,\omega) + Q \cdot \mathbf{R}(r_n,\omega) = \\ = \left[P \cdot \tilde{\mathbf{B}}(r_0, r_0, \omega) + Q \cdot \tilde{\mathbf{B}}(r_n, r_0, \omega)\right] \cdot \mathbf{C} = 0.$$

Because, $\tilde{\mathbf{B}}(r_0,r_0,\omega)=\mathbf{E},$ we come to equality

$$\left[\mathbf{P}+\mathbf{Q}\cdot\tilde{\mathbf{B}}(r_n,r_0,\omega)\right]\cdot\mathbf{C}=\mathbf{0}. \qquad (24)$$

For the existence of a non-trivial vector **C** in (24), necessary and sufficient condition is met

$$\det\left[\mathbf{P}+\mathbf{Q}\cdot\tilde{\mathbf{B}}(r_n,r_0,\omega)\right]=0. \qquad (25)$$

Let us denote $\tilde{\mathbf{B}}(r_n,r_0,\omega)\overset{df}{=}\begin{pmatrix} b_{11}(\omega) & b_{12}(\omega)\\ b_{21}(\omega) & b_{22}(\omega)\end{pmatrix}.$

Equation (25) is a characteristic equation of the eigenvalue problem (19, 20) which can be written in the expanded form as follows

$$r_0\alpha_0(r_n\alpha_n b_{12}(\omega)+b_{22}(\omega))+r_n\alpha_n b_{11}(\omega)+b_{21}(\omega)=0.$$

To find a non-trivial vector $\mathbf{C}=(C_1,\quad C_2)^T$ we put ω_k instead of ω into equality (24). So we come to vector equality

$$\left[\begin{pmatrix} r_0\alpha_0 & -1\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ r_n\alpha_n & 1\end{pmatrix}\cdot\begin{pmatrix} b_{11}(\omega_k) & b_{12}(\omega_k)\\ b_{21}(\omega_k) & b_{22}(\omega_k)\end{pmatrix}\right]\cdot\begin{pmatrix} C_1\\ C_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow$$

$$\Rightarrow\begin{pmatrix} \alpha_0 & -1\\ r_n\alpha_n b_{11}(\omega_k)+b_{21}(\omega_k) & r_n\alpha_n b_{12}(\omega_k)+b_{22}(\omega_k)\end{pmatrix}\cdot\begin{pmatrix} C_1\\ C_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

which is equivalent to a system of equations

$$\begin{cases} r_0\alpha_0 C_1-C_2=0\\ \left(r_n\alpha_n b_{11}(\omega_k)+b_{21}(\omega_k)\right)\cdot C_1+\\ \quad+\left(r_n\alpha_n b_{12}(\omega_k)+b_{22}(\omega_k)\right)\cdot C_2=0\end{cases}. \qquad (26)$$

Since the determinant of system (26) is zero, the system has non-trivial solutions $\mathbf{C}\neq 0\in\mathbb{R}$. Putting, for example $C_2=1$ we will get

$$\mathbf{C}=\left(\frac{1}{r_0\alpha_0},\quad 1\right)^T.$$

The eigenvectors of the differential system (21) with boundary conditions (8) have the form

$$\mathbf{R}_{ki}(r,\omega_k)=\tilde{\mathbf{B}}_i(r,r_i,\omega_k)\cdot\tilde{\mathbf{B}}(r_i,r_0,\omega_k)\cdot\begin{pmatrix}\dfrac{1}{\alpha_0 r_0}\\ 1\end{pmatrix},\quad i=\overline{0,n-1},$$

where the first coordinate is the eigenfunctions $R_k(r,\,\omega_k)$ and the second one is quasi-derivatives $R_k^{[1]}(r,\omega_k)$ of relevant eigenfunctions.

Development into Fourier series in its eigenfunctions $R_k(r,\,\omega_k)$. Let $g(r)=\sum\limits_{i=0}^{n-1}g_i(r)\theta_i$ be absolutely continuous on $[r_0,\,r_n]$ function, which has different analytical expressions $g_i(r)$ at each interval $[r_i,\,r_{i+1})$. Function development $g(r)$ in the Fourier series on its eigenfunctions $R_k(r,\,\omega_k)$ of tasks (19), (20) has an appearance

$$g(r)=\sum_{k=1}^{\infty}g_k R_k(r,\omega_k).$$

Where the Fourier coefficients g_k are calculated by the formula [8]

$$g_k=\frac{1}{\left\|R_k(r,\omega_k)\right\|^2}\sum_{i=0}^{n-1}c_i\rho_i\int_{r_i}^{r_{i+1}}r\cdot g_i(r)\cdot R_{ki}(r,\omega_k)dr,$$

and $\|R_k\|^2$ – the square of the norm of eigenfunctions $R_k(r,\,\omega_k)$

$$\left\|R_k(r,\omega_k)\right\|^2=\sum_{i=0}^{n-1}c_i\rho_i\int_{r_i}^{r_{i+1}}r\cdot R_{ki}^2(r,\omega_k)dr.$$

Building a mixed task solution for a function $v(r,\,\tau)$**.** The scheme of constructing the solution of this problem by the method of eigenfunctions is described in detail in [8]. This solution is represented as a vector function

$$\mathbf{v}_i(r,\tau)=\sum_{k=1}^{\infty}\left[f_k\cdot e^{-\omega_k\tau}-\int_0^{\tau}e^{-\omega_k(\tau-s)}u_k(s)ds\right]\cdot\mathbf{R}_{ki}(r,\omega_k), \qquad (27)$$

where f_k and u_k are coefficients of development of the initial condition $f(r)$ and of the function $\dfrac{\partial u}{\partial\tau}$ respectively into Fourier series by the system of eigenfunctions $R_k(r,\,\omega_k)$, the first coordinate is the desired function $v_i(r,\,\tau)$, and the second is its quasi-derivative $v_i^{[1]}(r,\tau)$.

Given (12) and (27), we obtain the solution of the original problem (1–3)

$$t(r,\tau)=\sum_{i=0}^{n-1}\left[u_i(r,\tau)+v_i(r,\tau)\right]\cdot\theta_i.$$

Model example. Consider the problem of heating a four-layer hollow cylindrical structure made of different isotropic layers. At the initial time, the temperature of the structure and the environment is 20 °C. The ambient temperature that washes the outside changes by law $\psi_n(\tau)=660(1-e^{-032\tau}-0.313e^{-3.8\tau})+20$. The ambient temperature in the middle of the structure is constant, and is 20 °C. Thermal characteristics of the design for calculation: radii of layers – $r_0=0.1$ m, $r_1=0.15$ m, $r_2=0.35$ m, $r_3=0.38$ m, $r_4=0.44$ m; coefficients of thermal conductivity – $\lambda_0=0.76$ W/m · °C, $\lambda_1=1.92$ W/m · °C, $\lambda_2=0.09$ W/m · °C, $\lambda_3=2.5$ W/m · °C; specific heat capacity – $c_0=870$ J/kg · °C, $c_1=550$ J/kg · °C, $c_2=1140$ J/kg · °C, $c_3=690$ J/kg · °C; density – $\rho_0=1800$ kg/m^3, $\rho_1=2500$ kg/m^3, $\rho_2=300$ kg/m^3, $\rho_3=1600$ kg/m^3; the intensity of the internal heat source $q_{v0}=960$ W/m^3, $q_{v2}=1150$ W/m^3; heat transfer coefficients – $\alpha_0=4$ W/m^2 · °C, $\alpha_n=25$ W/m^2 · °C.

The results of the calculations are shown in Fig. 1.

Consider the same structure which is heated to a temperature of 1100 °C and placed in an environment that washes the inner and outer surfaces of the structure with a temperature of 25 °C. The coefficients of heat exchange are $\alpha_0=\alpha_n=40$ W/m^2 · °C. The results of the calculations are shown in Fig. 2.

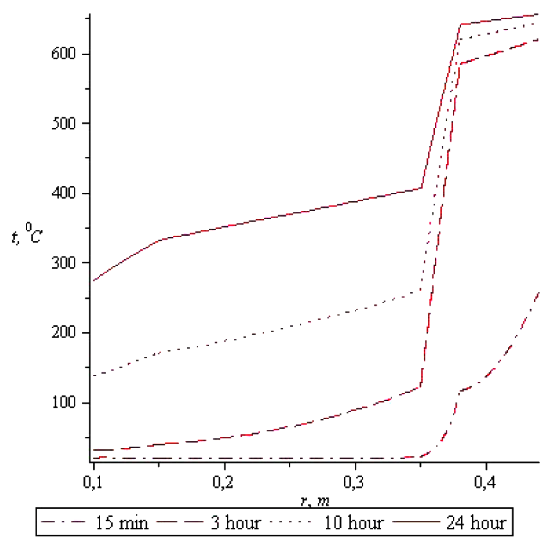

Fig. 1. Heating of a four-layer cylindrical structure

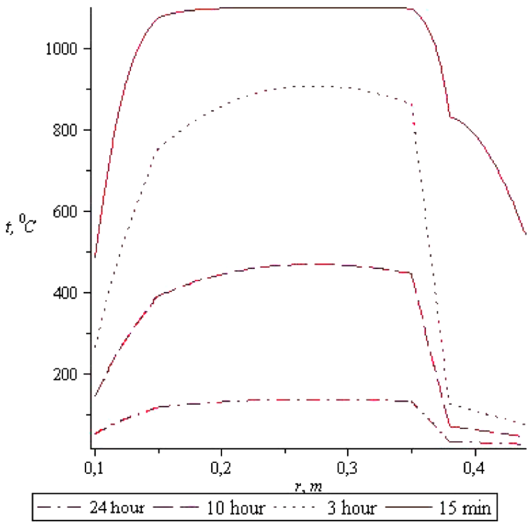

Fig. 2. Cooling of a four-layered cylindrical structure

The examples given are model examples that merely illustrate the possibilities of the proposed method.

Conclusions. The results obtained are directly applicable in a number of applications. Task (1) and (3) describe the processes of heat exchange (both heating and cooling) in multilayer hollow cylindrical structures, taking into account the boundary conditions of the third kind and the presence of internal heat sources. Changing the boundary conditions to any other (first or second kind) has absolutely no effect on the scheme of solving the task.

The system of equations (7) and (21) is in a class of absolutely continuous on $[r_0, r_n]$ vector functions, which meets the conditions of perfect thermal contact. In this connection, when setting the initial problem (1–3), there are no conjugation conditions (equality of temperature and heat fluxes).

References.

1. Kustov, V. V., Ropyak, L. Ya., Makoviychuk, N. V., & Ostapovich, V. V. (2016). Determination of the optimal allowances for machining of parts with coatings. *Metallurgical and Mining Industry, 1*, 164-171.

2. Ropyak, L. Ya., Shatskyi, I. P., & Makoviichuk, M. V. (2017). Influence of the Oxide-Layer Thickness on the Ceramic–Aluminium Coating Resistance to Indentation. *Metallofizika i noveishie tekhnologii, 39,* 517-524. https://doi.org/10.15407/mfint.39.04.0517.

3. Wojciki, W., Alimzhanova, Zh. M., Velyamov, T. T., & Akhmetova, A. M. (2019). About one model of pumping oil mixture of different viscosities through a single pipeline in an unsteady thermal field. *News of the National academy of sciences of the Republic of Kazakhstan. Series of Geology and Technical Sciences, 437,* 207-214. https://doi.org/10.32014/2019.2518-170X.144.

4. Eliseev, V. N., Tovstonog, V. A., & Borovkova, T. V. (2017). Solution algorithm of generalized non-stationary heat conduction problem in the bodies of simple geometric shapes *Herald of the Bauman Moscow State Technical University. Series Mechanical Engineering, 1*, 112-128. https://doi.org/10.18698/0236-3941-2017-1-112-128.

5. Colaço, M. J., Alves, C. J. S., & Bozzoli, F. (2015). The reciprocity function approach applied to the non-intrusive estimation of spatially varying internal heat transfer coefficients in ducts: numerical and experimental results. *International Journal of Heat and Mass Transfer, 90,* 1221-1231. https://doi.org/10.1016/j.ijheatmasstransfer.2015.07.028.

6. Daneshjou, K., Bakhtiari, M., Alibakhshi, R., & Fakoor, M. (2015). Transient thermal analysis in 2D orthotropic FG hollow cylinder with heat source. *International Journal of Heat and Mass Transfer, 89,* 977-984. https://doi.org/10.1016/j.ijheatmasstransfer.2015.05.104.

7. Yang, B., & Liu, S. (2017). Closed-form analytical solutions of transient heat conduction in hollow composite cylinders with any number of layers. *International Journal of Heat and Mass Transfer, 108,* 907-917. https://doi.org/10.1016/j.ijheatmasstransfer.2016.12.020.

8. Pazen, O. Yu., & Tatsii, R. M. (2017). Direct (classical) method of calculation of the temperature field in a hollow multilayer cylinder. *Journal of Engineering Physics and Thermophysics, 91,* 1373-1384. https://doi.org/10.1007/s10891-018-1871-3.

9. Tatsiy, R., Stasiuk, M., Pazen, O., & Vovk, S. (2018). Modeling of Boundary-Value Problems of Heat Conduction for Multilayered Hollow Cylinder. *Problems of Infocommunications. Science and Technology*, 21-25. https://doi.org/10.1109/INFOCOMMST.2018.8632131.

10. Shevelev, V. V. (2019). Stochastic Model of Heat Conduction with Heat Sources or Sinks. *Journal of Engineering Physics and Thermophysics, 91,* 614-624. https://doi.org/10.1007/s10891-019-01970-2.

Моделювання процесу теплообміну в багатошаровому порожнистому циліндрі з урахуванням внутрішніх джерел тепла

Р. М. Таццій, О. Ю. Пазен, С. Я. Вовк, Д. В. Харишин

Львівський державний університет безпеки життєдіяльності, м. Львів, Україна, e-mail: opazen@gmail.com

Мета. Встановити особливості розподілу нестаціонарного температурного поля за товщиною багатошарового порожнистого циліндра за умов конвективного теплообміну на його поверхнях з урахуванням наявності внутрішніх (розподілених) джерел тепла.

Методика. Задля досягнення поставленої мети було застосовано прямий метод розв'язування крайових задач теорії теплопровідності, що включає в себе застосування методу редукції, концепції квазіпохідних, методу відокремлення змінних і модифікованого методу власних функцій Фур'є.

Результати. Розв'язок поставленої задачі отримано в замкненому вигляді, що дозволило створити алгоритм розрахунку поширення нестаціонарного температурного поля в багатошарових порожнистих циліндричних конструкціях за умов конвективного теплообміну на його поверхнях і наявності внутрішніх джерел тепла. Варто відзначити, що до таких алгоритмів входять лише: а) знаходження коренів характеристичного рівняння; б) множення скінченної кількості відомих (2×2) матриць; в) обчислення визначених інтегралів; г) сумування необхідної кількості членів ряду для отримання заданої точності. Зміна крайових умов третього роду на будь-які інші крайові умови не викликає жодних істотних труднощів у розв'язку поставленої задачі.

Наукова новизна. Отримано розв'язок задачі про поширення нестаціонарного температурного поля в багатошаровому порожнистому циліндрі за наявності внутрішніх джерел тепла та конвективного теплообміну на його поверхнях.

Практична значимість. Упровадження результатів дослідження дає змогу досліджувати процеси нагрівання або охолодження багатошарових циліндричних порожнистих конструкцій з урахуванням внутрішніх джерел тепла, що зустрічаються у ряді прикладних задач. Це задачі, що можуть бути пов'язані із процесами охолодження тепловидільних елементів атомних електростанцій, зміни температурного поля при мікродуговому оксидуванні, нагрівання електронних компонентів при проходженні електричного струму.

Ключові слова: *теплопровідність, нестаціонарне температурне поле, порожнистий циліндр*

Моделирование процесса теплообмена в многослойном полом цилиндре с учетом внутренних источников тепла

Р. М. Таций, О. Ю. Пазен, С. Я. Вовк, Д. В. Харышин

Львовский государственный университет безопасности жизнедеятельности, г. Львов, Украина, e-mail: opazen@gmail.com

Цель. Установить особенности распределения нестационарного температурного поля по толщине многослойного полого цилиндра в условиях конвективного теплообмена на его поверхностях с учетом наличия внутренних (распределенных) источников тепла.

Методика. Для достижения поставленной цели был применен прямой метод решения краевых задач теории теплопроводности, который включает в себя применение метода редукции, концепции квазипроизводных, метода разделения переменных и модифицированного метода собственных функций Фурье.

Результаты. Решение поставленной задачи получено в замкнутом виде, что позволило создать алгоритм расчета распространения нестационарного температурного поля в многослойных полых цилиндрических конструкциях в условиях конвективного теплообмена на его поверхностях и наличии внутренних источников тепла. Стоит отметить, что в такие алгоритмы входит только: а) нахождение корней характеристического уравнения; б) умножение конечного числа известных (2×2) матриц; в) вычисление определенных интегралов; г) суммирование необходимого количества членов ряда для получения заданной точности. Изменение краевых условий третьего рода на любые другие краевые условия не вызывает никаких существенных трудностей в решении поставленной задачи.

Научная новизна. Получено решение задачи о распространении нестационарного температурного поля в многослойном полом цилиндре при наличии внутренних источников тепла и конвективного теплообмена на его поверхностях.

Практическая значимость. Внедрение результатов исследования позволяет исследовать процессы нагревания или охлаждения многослойных цилиндрических полых конструкций с учетом внутренних источников тепла, которые встречаются в ряде прикладных задач. Это задачи, которые могут быть связаны с процессами охлаждения тепловыделяющих элементов атомных электростанций, изменения температурного поля при микродуговом оксидировании, нагрева электронных компонентов при прохождении электрического тока.

Ключевые слова: *теплопроводность, нестационарное температурное поле, полый цилиндр*

Recommended for publication by M. M. Semerak, Doctor of Technical Sciences. The manuscript was submitted 08.09.19.