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OPTIMIZATION OF THE CAPITAL-LABOR RATIO ON THE BASIS OF PRODUCTION FUNCTIONS IN THE ECONOMIC MODEL OF PRODUCTION

Purpose. Development of a procedure for determining the optimal capital-labor ratio within the framework of two-factor production functions, as well as an analysis of the conditions for its implementation in practice.

Methodology. In the process of the conducted mathematical analysis, the search for the extrema of the most popular in economic studies, of two-factor production functions, namely the Cobb-Douglas production function, CES-function, linear function, Allen's function in terms of capital assets, was performed. The limit rate of the technological substitution of factors in conditions of optimal capital-labor ratio was found, which proved to be unique in all investigated production functions. The indicated relationship was verified on the basis of microeconomic theory, in particular, using the equilibrium principle.

Findings. The hypothesis of the single marginal rate of technological substitution in conditions of optimal capital-labor ratio is verified. On the basis of the microeconomic theory, the procedure for determining the optimal capital-labor ratio in the framework of any two-factor production functions is substantiated due to the marginal rate of technological substitution. The proposed procedure consists in determining the marginal rate of technological substitution as the ratio of the marginal products of the production factors and equating it to one unit. The obtained ratio makes it possible to deduce the formula of optimal capital-labor ratio, provided that the indicators of the dynamics of realized products, main assets and labor remuneration in the enterprise are adequately described by the corresponding production function with non-zero substitution.

Originality. Development of a procedure for optimization of capital-labor ratio within the framework of substitutional two-factor production functions based on the formulas of their marginal rate of technological substitution.

Practical value. Theoretical conclusions and proposals were tested on the example of a Ukrainian enterprise in the field of mechanical engineering. The results of the research can be used to manage capital-labor ratio by determining its optimal volume in production, which opens the possibility of rational use of basic productive assets and labor at domestic enterprises which are characterized by relative over-equipment with high moral and physical equipment decay.

Keywords: *optimal capital-labor ratio, production function, equivalence principle, marginal rate of technological substitution of factors*

Introduction. In the process of managing the market-production system (industry, enterprise, region), top managers always have the task of maintaining the best proportion in the distribution of capital to the costs associated with the main productive funds (K) and pay (L). The ratio K/L is known in the economy as capital-labor ratio – an indicator characterizing the degree of equipment availability, the magnitude of the main productive assets used by one employee or worker.

The growth of labor *capital-labor ratio* is considered to be one of the most important factors in increasing the efficiency of social production. On the basis of the introduction of innovation and investment measures, in particular, the automation of production processes, the technical re-equipment of enterprises, labor capital-labor ratio in the national economy of any state is steadily growing. This is a guarantee of high quality and competitiveness of products (works, services), compliance with European and world standards. From these perspectives, capital-labor ratio of Ukrainian enterprises due to the high decay of the main means of production is very far from the necessary, for example, the world capital-labor ratio.

The aforementioned statements largely reflect the essence of economic policy in the conditions of rigid centralized production management with a high proportion of manual and weakly mechanized labor, when measures to introduce the latest technology and technology were accompanied by an automatic increase in demand for additional output. If the transitional market economy, like the Ukrainian one, is studied, then the relationship between capitalization and volume of implementation is less clear and functional. In this situation, the growth of the K/L ratio may not correlate with changes in demand for additional products, since the amount of net sales revenue largely depends on the solvency of consumers of these products. At the same time, an important factor in the demand for additional products is the growth of real wages in the country.

Therefore, the change in the capital-labor ratio of the market-production system should be considered and analyzed in close association with the dynamics of its volume of output (Y). The fact is that excessive growth of the K/L ratio usually results in deterioration in the use of core production assets and, as a consequence, a decrease in return on capital Y/K. The opposite situation arises in the case of unjustified decline in capital-labor ratio which inevitably leads to a decrease in labor productivity Y/L.

We think, that it is better to talk about the optimal capital-labor ratio in terms of maximizing the volume of output at given total capital costs (or minimizing total capital costs for a given volume of output). This statement is based on the fundamental provision of the microeconomy, according to which the optimum of the commodity producer is achieved at the point where the weighted by price marginal products of the production factors coincide. Realizing this condition, the enterprise as a commodity producer reaches the state of internal equilibrium, that is, the best combination of resources K, L. As is known, the marginal categories have found their full implementation in the theory of marginalism, and in particular, in the mathematical economics branch of economic science – the theory of production functions.

Literature review. In the domestic literature, insufficient attention to the indicator of capital-labor ratio within the framework of the theory of the production function has been paid. Usually, authors studied the K/L ratio as a certain derivative value only, which can be calculated within the framework of the analyzed production function.

Indeed, using simple transformations, the original production function was shown when the capital-labor ratio served as a new factor. For example, if the output of production function had the form $Y=f(K, L)$, and dividing its left and right sides by L, the following was obtained $Y/L = f(K, L, K/L)$, that is, the function that reflects the dependence of labor productivity on three factors, among which was the capital-labor ratio.

The analysis of foreign sources of recent years has shown that the approach to studying the K/L indicator, which potentially can and should be optimized on the basis of production function, is also completely absent. Thus, in the works [1, 2] the issues of determining the nature and conditions for maximizing neoclassical production functions are discussed but an assessment of the impact of the level of capital-labor ratio on output is not given. The problems of studying the elasticity of the substitution of capital invested in the main productive funds and labor were discussed by the authors of the work [3], who analyzed the CES function (Constant Elasticity of Substitution). They justified the fact that higher elasticity of substitution could lead to a higher level of capital-labor ratio with a maximization of output, which is crucial for economic growth.

In the work [4], a two-factor production function with a known elasticity of labor substitution by capital was studied to assess changes in the capital-labor ratio when the marginal rate of substitution changes. The publication [5] is devoted to the study of the advantages and disadvantages of the CES-function and the Cobb-Douglas production function in their use in forecasting production.

In the work [6] the author studies the methodological questions of the application of the Cobb-Douglass production function in the problems of forecasting the long-run equilibrium level of production, depending on two factors: labor and capital. The research [7] highlights the most dominant problem in evaluating the parameters of the CES-function, which is related to the local minima of the target function. The authors proposed ways to overcome the problems of constructing the CES function, since it is less restrictive with respect to the interaction of the factors K and L in comparison with the Cobb-Douglas production function.

A certain breakthrough in the study of the index of capital-labor ratio on the basis of two-factor production functions was observed at Odessa National Economic University with the appearing of scientific developments in this direction [8, 9], in which for the first time the optimum of a commodity producer was determined in the case of the use of Cobb-Douglas production function in its classical and dynamical (taking into account the time factor) variants.

Unsolved aspects of the problem. We think that the problem of finding the optimal capital-labor ratio according to the criterion of maximum output with the given total capital costs is highly relevant and little studied. In particular, it is of definite theoretical and practical interest to determine the best K/L ratio for such two-factor production functions as the CES-function, the Cobb-Douglas function, the linear function, the Allen's function, etc.

Purpose. The purpose of the study was to develop a procedure for determining the optimal capital-labor ratio in the framework of the two-factor production function, as well as an analysis of the conditions for its implementation in practice. To achieve the set goal, the following tasks were set: 1) to search for extrema of the most popular production functions in terms of K/L on the basis of mathematical analysis; 2) to test the hypothesis of a single marginal rate of substitution of substitutional production functions in conditions of optimal capital-labor ratio; 3) to justify the procedure for determining the optimal ratio of $K1/L1$ in the framework of various two-factor production functions using the marginal rate of substitution; 4) to analyze the growth trajectory reflecting the effect of the scale of production on the basis of the developed procedure for determining the optimal capital-labor ratio within the framework of various two-factor production functions; 5) to test the theoretical results of the study on the example of statistical data of domestic commodity producers.

Results. Determination of the extrema of the most popular production functions by the magnitude of capital-labor ratio K/L on the basis of mathematical analysis is to be started with the function of Cobb-Douglas

$$Y = AK^{\alpha}L^{\beta}, \tag{1}$$

where A is the coefficient of scale ($0 < A$); α, β are unknown parameters characterizing the elasticity of output by production factors ($0 < \alpha < 1$, $0 < \beta < 1$).

It is then assumed that all variables of the models Y, K, L are presented in cost terms.

As shown in the works [8, 9], for a given capital cost $C = K + L$, the production function (1) reaches its maximum

$$Y_{MAX} = A\left(\frac{\alpha}{\beta}\right)^{\alpha} L_1^{\alpha+\beta} = A\left(\frac{\beta}{\alpha}\right)^{\beta} K_1^{\alpha+\beta}, \tag{2}$$

at

$$K_1 = \frac{\alpha}{\alpha+\beta}C_1; \quad L = \frac{\beta}{\alpha+\beta}C_1.$$

Where the optimal capital-labor ratio is equal to

$$K_1/L_1 = \alpha/\beta. \tag{3}$$

Consequently, investing the capital in the main production funds and labor in proportion (3), provided that the relationship between production output and two aggregated production factors is adequately described by the Cobb-Douglas production function, the commodity producer receives the largest output (2), or provides a minimum total cost capital

$$C_{MIN} = \frac{\alpha+\beta}{\alpha}K_1 = \frac{\alpha+\beta}{\beta}L_1.$$

The function with constant elasticity of resource substitution or the CES function is the second most popular production function in economic research

$$Y = A_0[A_1K^{-p} + (1-A_1)L^{-p}]^{-\frac{\gamma}{\partial}}, \tag{4}$$

where A_0 is the coefficient of scale ($0 < A_0$); A_1 is the weight factor of the production factor ($0 < A_1 < 1$); p is the coefficient of substitution ($-1 < p$); γ is the homogeneity index ($0 < \gamma$).

Searching for extrema and deducing the formula of optimal capital for production functions (4) are given in the works [10, 11]

$$\begin{aligned} Y_{MAX} &= A_0L_1^{\gamma}[(1-A_1)(K_1/L_1+1)]^{-\frac{\gamma}{p}} = \\ &= A_0K_1^{\gamma}[A_1(1+L_1/K_1)]^{-\frac{\gamma}{p}}, \end{aligned} \tag{5}$$

at

$$K_1 = \frac{\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}}{1+\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}}C_1; \quad L_1 = \frac{1}{1+\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}}C_1. \tag{6}$$

From formulas (6) the optimal capital-labor ratio equals

$$K_1/L_1 = \left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+\partial}}. \quad (7)$$

Thus, applying the proportion (7), provided that the relationship between production output and aggregated production factors is adequately described by the CES function, the commodity producer maximizes output (5), or provides the minimum total cost of capital

$$C_{MIN} = \left[\frac{1+\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}}{\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}}\right] K_1 = \left[1+\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}\right] L_1.$$

Let the activity of a market-and-production system be precisely approximated by a linear function

$$Y = A_2K + A_3L, \quad (8)$$

where A_2 , A_3 are coefficients at production factors that characterize their marginal products.

Condition of maximizing output at a given total cost of capital $C = K + L$ is

$$Y = A_2K + A_3(C - K) \to \max. \quad (9)$$

We find the first partial derivative of expression (9) for a variable K and equate it to zero

$$\frac{\partial Y}{\partial K} = A_2 - A_3 = 0. \quad (10)$$

Expression (10) shows that the maximum output for a linear production function does not depend on the selected values of K, L, but the condition must be fulfilled $A_2 = A_3 = a$. If it is fulfilled, then any point on the line $Y = a(K + L)$ will satisfy the condition (9) and the task of determining the optimal capital-labor ratio has an infinite set of solutions. At the same time, the maximum of producer's products is equal $Y_{MAX} = aC_1$, and the minimum total capital costs are $C_{MIN} = Y/a$.

Let the activity of the market-production system be quite accurately approximated using the Allen's production functions [11]

$$Y = A_4KL - A_5K_2 - A_6L^2, \quad (11)$$

where A_4, A_5, A_6 are unknown coefficients ($0 < A_4, A_5, A_6$).

The condition of maximizing output in the given total capital costs $C = K + L$ is

$$\begin{aligned} Y &= A_4K(C - K) - A_5K^2 - A_6(C - K)^2 = \\ &= -K^2(A_4 + A_5 + A_6) + CK(A_4 + 2A_6) - A_6C^2 \to \max. \end{aligned} \quad (12)$$

We find the first partial derivative of expression (12) for the variable K and equate it to zero

$$\frac{\partial Y}{\partial K} = -2(A_4 + A_5 + A_6)K + C(A_4 + 2A_6) = 0.$$

From here, we get critical points

$$K_1 = \frac{A_4 + 2A_6}{2(A_4 + A_5 + A_6)}C_1; \quad L_1 = \frac{A_4 + 2A_5}{2(A_4 + A_5 + A_6)}C_1, \quad (13)$$

and optimal capital-labor ratio

$$K_1/L_1 = \frac{A_4 + 2A_6}{A_4 + 2A_5}. \quad (14)$$

Substituting the values of K_1, L_1 from the formula (13) in the Allen's production function, we determine the maximum output in terms of optimal capital-labor ratio

$$\begin{aligned} Y_{MAX} &= A_4K_1C_1\frac{A_4 + 2A_5}{2(A_4 + A_5 + A_6)} - A_5K_1^2 - \\ &- A_6C_1^2\left[\frac{A_4 + 2A_5}{2(A_4 + A_5 + A_6)}\right]^2 = A_4L_1C_1\frac{A_4 + 2A_6}{2(A_4 + A_5 + A_6)} - \\ &- A_6L_1^2 - A_5C_1^2\left[\frac{A_4 + 2A_6}{2(A_4 + A_5 + A_6)}\right]^2. \end{aligned} \quad (15)$$

Thus, investing the capital in the main production funds and payment in proportion (14), provided that the relationship between production output and two aggregated production factors is adequately described by Allen's production function, the commodity producer receives the largest output (15), or provides the minimum total cost of capital.

$$C_{MIN} = \frac{2(A_4 + A_5 + A_6)}{A_4 + 2A_6}K_1 = \frac{2(A_4 + A_5 + A_6)}{A_4 + 2A_5}L_1.$$

Now we test the hypothesis of a single *MRTS* (Marginal Rate of Technological Substitution) in terms of optimal capital-labor ratio of the commodity producer. For this we will consistently substitute the expression K_1/L_1 for the above-discussed production function in the formula $MRTS_{LK}$

$$MRTS_{LK} = \frac{\partial Y}{\partial L} : \frac{\partial Y}{\partial K} = \frac{f_L'(K,L)}{f_K'(K,L)}, \quad (16)$$

where $f_K'(K, L)$, $f_L'(K, L)$ are the first partial derivatives of the corresponding V production functions with variables K, L expressed in monetary units.

For the Cobb-Douglas production functions the value (16) takes the form

$$MRTS_{LK} = \frac{\partial Y}{\partial L} : \frac{\partial Y}{\partial K} = \frac{f_L'(K,L)}{f_K'(K,L)} = \frac{A\beta K^{\beta}L^{\beta-1}}{A\alpha K^{\alpha-1}L^{\beta}} = \frac{\beta}{\alpha} \times \frac{K}{L}. \quad (17)$$

Substituting (17) expression (3), we obtain

$$MRTS_{LK} = \frac{\beta}{\alpha} \times \frac{K_1}{L_1} = \frac{\beta}{\alpha} \times \frac{\alpha}{\beta} = 1,$$

and the tested hypothesis is confirmed.

Similarly, for the CES function

$$\begin{aligned} MRTS_{LK} &= \frac{\partial Y}{\partial L} : \frac{\partial Y}{\partial K} = \frac{f_L'(K,L)}{f_K'(K,L)} = \\ &= \frac{\gamma(1-A_1)Y^{1+\frac{p}{\gamma}}}{L^{1+p}A_0^{\frac{p}{\gamma}}} : \frac{\gamma A_1 Y^{1+\frac{p}{\gamma}}}{K^{1+p}A_0^{\frac{p}{\gamma}}} = \frac{1-A_1}{A_1}\left(\frac{K}{L}\right)^{1+p}. \end{aligned} \quad (18)$$

Substituting expression (7) in (18), we obtain

$$MRTS_{LK} = \frac{1-A_1}{A_1}\left(\frac{K_1}{L_1}\right)^{1+p} = \frac{1-A_1}{A_1}\left[\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}\right]^{1+p} = 1.$$

Again the hypothesis is confirmed.

For the linear function

$$MRTS_{LK} = \frac{\partial Y}{\partial L} : \frac{\partial Y}{\partial K} = \frac{f_L'(K,L)}{f_K'(K,L)} = \frac{A_3}{A_2}. \quad (19)$$

As shown above (see formula (10)), the condition for obtaining the maximum output is equal to $A_2 = A_3$. Therefore, it is obvious that, at optimum capital-labor ratio of the commodity producer, the expression (19) is equal to one and the hypothesis that is being tested is valid.

For Allen's production function

$$MRTS_{LK} = \frac{\partial Y}{\partial L} : \frac{\partial Y}{\partial K} = \frac{f'_L(K,L)}{f'_K(K,L)} = \frac{A_4 - 2A_5\left(\dfrac{K}{L}\right)}{A_4\left(\dfrac{K}{L}\right) - 2A_6}. \quad (20)$$

Substituting expression (14) in (20), we obtain

$$MRTS_{LK} = \frac{A_4 - 2A_5\left(\dfrac{K}{L}\right)}{A_4\left(\dfrac{K}{L}\right) - 2A_6} = \frac{A_4 - 2A_5\dfrac{A_4 + 2A_6}{A_4 + 2A_5}}{A_4\dfrac{A_4 + 2A_6}{A_4 + 2A_5} - 2A_6} = \\ = \frac{A_4^2 - 4A_5A_6}{A_4^2 - 4A_5A_6} = 1.$$

Thus, for the four substitutional production functions that were examined the assumption of a single marginal rate of technological substitution of the *MRTS* in terms of optimal capital-labor ratio was confirmed. Now we will summarize the results obtained and justify the procedure for determining the optimal ratio K_1/L_1 within the framework of various substitutional two-factor production functions using the marginal rate of technological substitution. To this end, we will use the well-known provisions of the microeconomic theory about the requirements for the optimum of the commodity producer, which ensure the internal equilibrium of the market-production system [12].

Requirement 1. The ratio of the marginal products of the production factors MP_K, MP_L should be equal to the ratio of their average prices p_K, p_L

$$\frac{MP_K}{MP_L} = \frac{p_K}{p_L}. \quad (21)$$

Requirement 2. Marginal products of production factors per 1 monetary unit must be the same

$$\frac{MP_K}{p_K} = \frac{MP_L}{p_L}. \quad (22)$$

Requirement (22) is often referred to the equivalence principle – price-weighted marginal products of production factors should be aligned. Implementing these requirements, the market-production system achieves a state of internal equilibrium, that is, a better combination of resources *K, L*, which means – the optimal capital-labor ratio.

Obviously, in the case of measuring all the variables *Y, K, L* in monetary units, the marginal products coincide, and the corresponding marginal rates of technological substitution $MRTS_{LK}$ are equal to one.

Consequently, we come to the following conclusion: in order to determine the coordinates of the optimal capital-labor ratio on the basis of two-factorial production function, it is sufficient to find the expression of its marginal rate of technological substitution and equate it to one unit. In other words, the definition of the optimal capital-labor ratio in the framework of two-factor production function consists in expressing the ratio K_1/L_1 provided

$$MRTS_{LK} = \frac{\partial Y}{\partial L} : \frac{\partial Y}{\partial K} = \frac{f'_L(K,L)}{f'_K(K,L)} = 1. \quad (23)$$

Obviously, in the long run, a simultaneous change in the production factors *K* and *L* may occur with an increase in the volume of output *Y*. This process is graphically shown by the growth of isocost with appropriate movement of isoquants with corresponding equilibrium points of the producer, which are connected by a curve – a growth trajectory.

The growth trajectory is an isocline for which requirement (23) should be applied. It combines the equilibrium points of the commodity producer in accordance with the increase in total capital expenditures $C = K + L$ with an increase in production volumes *Y*. Its form depends on the efficiency of increasing output volumes and reflects the effect of scale production.

In the case when enterprises produce homogeneous or conditionally homogeneous products, for example, coal mines, cement plants, etc., some production functions variables, for example, *Y, L*, can be measured in physical units.

Then from formula (21) it follows that expression (23) takes the following form

$$MRTS_{LK} = \frac{MP_L}{MP_L} = \frac{\partial Y}{\partial K} : \frac{\partial Y}{\partial L} = \frac{f'_K(K,L)}{f'_L(K,L)} = \frac{p_K}{p_L}. \quad (24)$$

In this situation, the growth trajectory will be determined by the isocline, for which the requirement (24) should be met.

From formulas (23, 24), the optimal capital-labor ratio of $K_1/L_1$is quite easy to find.

Approbation of the theoretical results obtained is completed using the example of statistical reporting data of machine-building LLC "ZHYTOMYRTEPLOMASH" for 2010–2017, which produces radiators and central heating boilers (Table 1).

Here, *Y* is the net proceeds from sales of the enterprise's products, *K* is the residual value of the main production assets, *L* is labor costs, *t* is time ($t = 1, 2, ..., N$), *k* is thousand.

Since the empirical data is represented by time series, in the process of modeling, dynamic production functions were used to take into account the influence on *Y* of all factors, except *K* and *L*, embodied in the time factor *t*. In the course of a mathematical statistical study, various dynamics of the production factors were tested. Their analysis showed insufficient reliability of the found parameters of production models based on the Cobb-Douglas, linear and Allen's functions due to a small sample ($N = 8$).

Highly accurate and statistically significant was the only dynamic CES function, which was created according to the data of Table 1 based on the use of an iterative algorithm for minimizing the target function of the remainders of the model by the Marquardt method. At the last iteration, the optimal solution was obtained (Table 2).

Thus, the desired model will be written as follows

$$Y = 9.89579e^{-0.06961t}[0.02230K^{-0.56244} + \\ + 0.97770L^{-0.56244}]^{-1.77795}. \quad (25)$$

Here, the substitution parameter p is found from the relationship: $p = 1/\sigma_{LK} - 1 = 0.56244$.

Table 1

Output data for the modeling of the dependence of output of "ZHYTOMYRTEPLOMASH" LLC on the costs of aggregated production factors

Years	*Y*, *k*, UAH	*K*, *k*, UAH	*L*, *k*, UAH	*t*
2010	66087	5226	7059	1
2011	103 704	5398	10 070	2
2012	93 397	5346	11 373	3
2013	90 837	4986	12 675	4
2014	116 358	5769	16 415	5
2015	124 762	5371	17 450	6
2016	88 978	6017	13 778	7
2017	132 969	6162	23 150	8

Table 2

Results of statistical modeling of the CES function based on data of "ZHYTOMYRTEPLOMASH" LLC

Constant A_0	9.89579
Scientific and technological progress ω	–0.06961
Elasticity of replacement of resources σ_{LK}	0.64002
Parameter of distribution A_1	0.02230
Marquardt number	0.01000
Adjusted coefficient of determination R^2	0.99999
The sum of squares of regression residues *RSS*	0.03096
Durbin-Watson coefficient *DW*	2.01023

The equation (25) with a very high accuracy describes the dynamics of net revenue from sales of products of Zhytomyrteplomash for the period under consideration. The coefficient of determination indicates that 99.9 % of the variation in revenue from sales of products is explained by the dynamic CES function, and the absolute error of the model which is 3.1 %. The Durbin-Watson criterion DW = 2.01 indicates a high adequacy of the constructed production function (optimal value 2.0). The growth rate of the neutral scientific and technical progress ω = –0.6961 shows that the company's average annual net revenues from sales decreased by almost 7 % under the influence of all factors, except for changes in capital expenses on basic production funds and labor.

Based on the data of Table 1, we calculate the marginal rate of technological substitution of $MRTS_{LK}$ for each year of the investigated period.

$$\frac{1-A_1}{A_1}=\frac{1-0.0223}{0.0223}=43.84756;\quad 1+p=1+0.56244=1.56244.$$

The results of calculating the marginal rate of substitution of $MRTS_{LK}$ are presented in Table. 3

In this case, all values of the marginal rate of technological substitution of resources exceed one unit (see Table 3), that is, there is a case where the actual capital-labor ratio significantly exceeds the optimal one. However, it should be noted that the investigated enterprise has a positive tendency to reduce the value of $MRTS_{LK}$ and gradually bring it closer to the optimal level.

Based on the parameters of the model (25), we calculate the average index of optimal capital-labor ratio for the enterprise "ZHYTOMYRTEPLOMASH" LLC by the formula (7)

Table 3

Actual capital-labor ratio and marginal rate technological substitution of "ZHYTOMYRTEPLOMASH" LLC

Years	Capital-labor ratio $\frac{K}{L}$, UAH /UAH	Marginal rate technological substitution $MRTS_{LK}$
2010	0.74033	27.41143
2011	0.53605	16.55168
2012	0.47006	13.48049
2013	0.39337	10.20590
2014	0.35145	8.55812
2015	0.30779	6.95636
2016	0.43671	12.01628
2017	0.26618	5.54382

$$\frac{K_1}{L_1}=\left(\frac{A_1}{1-A_1}\right)^{\frac{1}{1+p}}=\left(\frac{0.0223}{1-0.0223}\right)^{0.64003}=0.08894.$$

If we refer to the data in the second column of the Table 3, we can see that the actual capital-labor ratio in the enterprise substantially exceeds the optimal. This means that the production funds of "ZHYTOMYRTEPLOMASH" LLC are in a certain excess. This conclusion was confirmed also by the Cobb-Douglas production function: the coefficient of the variable $\ln(K)$ was statistically insignificant, unreliable (the Student's t-test is 0.705, the p-significance is 0.519).

The closest value to the optimal one was the actual value of $MRTS_{LK}$= 5.544, or capital-labor ratio of 0.26618 UAH/UAH. They were observed in 2017 and this year the company really received the maximum net sales revenue in the amount of 132.969 thousand UAH for the studied period (Table 1).

In this regard, a question arises: what net revenue from the sale of products would "ZHYTOMYRTEPLOMASH" LLC receive in 2017 with an optimal capital ratio of 0.08894? To answer it, we will use the calculation of the required volume of the main production funds in 2017 under optimal capital-labor ratio

$$K_{2017}=\frac{K_1}{L_1}\cdot L_{2017}=0.08894\cdot 23\,150=2058.96.$$

Thus, it would be enough to limit with the use of the main production funds in the amount of 2059 thousand UAH. At the same time, the net revenue from the sale of products of the enterprise would remain unchanged.

Consequently, the reserve for the growth of net revenues from the sale of products at the expense of optimization of capital-labor ratio at the enterprise – in this case as a result of the sale of part of unused basic production funds, in 2017 was approximately 4103 thousand UAH (6162 – 2059). Hence, the revenue from the sale of production of Zhytomyrteplomash LLC in 2017 under the optimal capital-labor ratio, would be 137 072 thousand UAH. (132 969 + 4103).

Thus, the given example confirms that in the industrial sectors of Ukraine in the years studied there was a slight excess of the basic production funds compared with the wages [13].

The actual excess of the required capital-labor ratio during this period, that is its non-optimality, was formed under the influence of two main factors:

1) the presence of a large amount of obsolete and unused technological equipment at enterprises of the domestic industry;

2) artificial underestimating of the productive factor of "labor force", which manifests itself in unreasonably low wages at the enterprises under study.

Conclusions. As the advantage of the study, we consider the fact that the problem of managing the capital-labor ratio, the need to find its optimal value in production is under the study of a wide range of domestic and foreign economists and practitioners. Its relevance and importance for the economy of any state, including Ukraine, is substantiated. For the first time, a procedure has been proposed for determining the optimal capital-labor ratio on the basis of substitutional production functions, whose theoretical foundation is the equivalence principle of microeconomics. Theoretical conclusions and proposals were tested on a specific example of the Ukrainian commodity producer.

As prospects for further developments of this direction, it is necessary to study optimal capital-labor ratio, as well as the optimal trajectories of the development of enterprises within other production functions such as the functions of Solow, Leontiev's functions, etc.

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Оптимізація фондоозброєності на основі виробничих функцій в економічній моделі виробництва

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Мета. Розробка процедури визначення оптимальної фондоозброєності в рамках двофакторних виробничих функцій, а також аналіз умов її реалізації на практиці.

Методика. У процесі проведеного математичного аналізу був здійснений пошук екстремумів найбільш популярних в економічних дослідженнях двофакторних виробничих функцій – Кобба–Дугласа, CES-функції, лінійної функції, функції Аллена за величиною фондоозброєності. Знайдена гранична норма технологічного заміщення факторів в умовах оптимальної фондоозброєності, що виявилася в усіх досліджуваних виробничих функціях одиничною. Указане співвідношення було перевірене на основі мікроекономічної теорії, зокрема, за допомогою еквімаржинального принципу.

Результати. Перевірена гіпотеза одиничної граничної норми технологічного заміщення в умовах оптимальної фондоозброєності. На основі мікроекономічної теорії обгрунтована процедура визначення оптимальної фондоозброєності в рамках будь-яких двофакторних виробничих функцій за допомогою граничної норми технологічного заміщення. Запропонована процедура полягає у визначенні граничної норми технологічного заміщення як відношення граничних продуктів виробничих факторів і прирівнювання його до одиниці. Отримане співвідношення дозволяє досить просто вивести формулу оптимальної фондоозброєності за умови, що показники динаміки реалізованої продукції, основних фондів і оплати праці на підприємстві адекватно описуються відповідною виробничою функцією з ненульовим заміщенням.

Наукова новизна. Полягає в розробці процедури оптимізації фондоозброєності в рамках субституційних двофакторних виробничих функцій на базі формул їх граничної норми технологічного заміщення.

Практична значимість. Теоретичні висновки та пропозиції були апробовані на прикладі українського підприємства в галузі машинобудування. Результати дослідження можуть бути використані для управління фондоозброєністю через визначення її оптимальної величини на виробництві, що відкриває можливість раціонального використання основних виробничих фондів і робочої сили на вітчизняних підприємствах, для яких характерно відносне надлишкове устаткування з високим моральним і фізичним зносом.

Ключові слова: *оптимальна фондоозброєність, виробнича функція, еквімаржинальний принцип, гранична норма технологічного заміщення факторів*

Оптимизация фондовооруженности на основе производственных функций в экономической модели производства

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Цель. Разработка процедуры определения оптимальной фондовооруженности в рамках двухфакторных производственных функций, а также анализ условий ее реализации на практике.

Методика. В процессе проведенного математического анализа был осуществлен поиск экстремумов наиболее популярных в экономических исследованиях двухфакторных производственных функций – Кобба–Дугласа, CES-функции, линейной функции, функции Аллена по величине фондовооруженности. Найдена предельная норма технологического замещения факторов в условиях оптимальной фондовооруженности, которая оказалась во всех исследуемых производственных функциях единичной. Указанное соотношение было проверено на основе микроэкономической теории, в частности, с помощью эквимаржинального принципа.

Результаты. Проверена гипотеза единичной предельной нормы технологического замещения в условиях оптимальной фондовооруженности. На основе микроэкономической теории обоснована процедура определения оптимальной фондовооруженности в рамках любых двухфакторных производственных функций с помощью предельной нормы технологического замещения. Предложенная процедура заключается в определении предельной нормы технологического замещения как отношения предельных продуктов факторов производства и приравнивание его к единице. Полученное соотношение позволяет достаточно просто вывести формулу оптимальной фондовооруженности при условии, что показатели динамики реализованной продукции, основных фондов и оплаты труда на предприятии адекватно описываются соответствующей производственной функцией с ненулевым замещением.

Научная новизна. Заключается в разработке процедуры оптимизации фондовооруженности в рамках субституциональной двухфакторной производственной функции на базе формул их предельной нормы технологического замещения.

Практическая значимость. Теоретические выводы и предложения были апробированы на примере украинского предприятия в отрасли машиностроения. Результаты исследования могут быть использованы для управления фондовооруженностью через определение ее оптимальной величины на производстве, что предоставляет возможность рационального использования основных производственных фондов и рабочей силы на отечественных предприятиях, для которых характерно относительное избыточное оборудование с высоким моральным и физическим износом.

Ключевые слова: *оптимальная фондовооруженность, производственная функция, эквимаржинальный принцип, предельная норма технологического замещения факторов*

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